Problem 12103
(American Mathematical Monthly, Vol.126, March 2019)

Proposed by G. Apostolopoulos (Greece).

Let a, b, and c be the side lengths of a triangle with inradius r and circumradius R. Let r_a, r_b, and r_c be the exradii opposite the sides of length a, b, and c, respectively. Prove

$$\frac{1}{2R^3} \leq \frac{r_a}{a^4} + \frac{r_b}{b^4} + \frac{r_c}{c^4} \leq \frac{1}{16r^3}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Let s be the semiperimeter of the triangle and let A be its area. Then we have that

$$r = \frac{A}{s},\ r_a = \frac{A}{s-a},\ r_b = \frac{A}{s-b},\ r_c = \frac{A}{s-c},\ R = \frac{abc}{4A}$$

and the inequalities can be written as

$$\frac{2^5 A^2}{(abc)^3} \leq \frac{1}{a^4(s-a)} + \frac{1}{b^4(s-b)} + \frac{1}{c^4(s-c)} \leq \frac{s^3}{16A^4}.$$

Let $x = (b+c-a)/2 > 0$, $y = (c+a-b)/2 > 0$, and $z = (a+b-c)/2 > 0$ then

$$a = x+y,\ b = y+z,\ c = z+x,\ s = x+y+z,\ A^2 = xyz(x+y+z)$$

and the inequalities become

$$\frac{32(xyz)(x+y+z)}{((x+y)(y+z)(z+x))^3} \leq \frac{1}{z(x+y)^4} + \frac{1}{x(y+z)^4} + \frac{1}{y(z+x)^4} \leq \frac{(x+y+z)}{16(xyz)^2}$$

or

$$\frac{2^5(xyz)^3}{((x+y)(y+z)(z+x))^3} \leq \sum_{\text{cyc}} \frac{z}{x+y+z} \cdot \frac{x^2y^2}{(x+y)^4} \leq \frac{1}{16}.$$

The inequality on the right holds because $(x+y)/2 \geq \sqrt{xy}$ implies

$$\sum_{\text{cyc}} \frac{z}{x+y+z} \cdot \frac{x^2y^2}{(x+y)^4} \leq \sum_{\text{cyc}} \frac{z}{x+y+z} \cdot \frac{1}{2^4} = \frac{1}{16}.$$

As regards the inequality on the left, by AM-GM inequality,

$$\frac{3(xyz)^{5/3}}{(x+y+z)((x+y)(y+z)(z+x))^{4/3}} \leq \sum_{\text{cyc}} \frac{z}{x+y+z} \cdot \frac{x^2y^2}{(x+y)^4}.$$

Hence it remains to show that

$$\frac{2^3}{3^{3/5}}(x+y+z)^{3/5}(xyz)^{4/5} \leq (x+y)(y+z)(z+x).$$

Since $(x+y)/2 \geq \sqrt{xy}$, we have that

$$\begin{aligned}(x+y)(y+z)(z+x) &= (x+y+z)(xy+yz+xy) - xyz \\ &\geq (x+y+z)(xy+yz+xy) - \frac{1}{8}(x+y)(y+z)(z+x).\end{aligned}$$

Therefore, by AM-GM inequality, it follows that

$$\begin{aligned}(x+y)(y+z)(z+x) &\geq \frac{8}{9}(x+y+z)(xy+yz+xy) \\ &= \frac{8}{9}(x+y+z)^{3/5}(x+y+z)^{2/5}(xy+yz+xy) \\ &\geq \frac{2^3}{3^2}(x+y+z)^{3/5} \cdot 3^{2/5}(xyz)^{2/15} \cdot 3(xyz)^{2/3} \\ &= \frac{2^3}{3^{3/5}}(x+y+z)^{3/5}(xyz)^{4/5}\end{aligned}$$

and we are done. □