Problem 12102
(American Mathematical Monthly, Vol.126, March 2019)

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Prove

$$\sum_{n=1}^{\infty} H_n^2 \left(\zeta(2) - \sum_{k=1}^{n} \frac{1}{k^2} - \frac{1}{n}\right) = 2 - \zeta(2) - 2\zeta(3)$$

where $H_n = \sum_{j=1}^{n} \frac{1}{j}$ is the n-th harmonic number.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. For $N \geq 1$

$$\begin{aligned} S_N := \sum_{n=1}^{N} H_n^2 \left(\zeta(2) - \sum_{k=1}^{n} \frac{1}{k^2} - \frac{1}{n}\right) &= \sum_{n=1}^{N} H_n^2 \sum_{k=n+1}^{\infty} \frac{1}{k^2} - \sum_{n=1}^{N} \frac{H_n^2}{n} \\ &= \sum_{k=1}^{N} \frac{1}{k^2} \sum_{n=1}^{k-1} H_n^2 - \sum_{n=1}^{N} \frac{H_n^2}{n} + \left(\zeta(2) - \sum_{k=1}^{N} \frac{1}{k^2}\right) \cdot \sum_{n=1}^{N} H_n^2. \end{aligned}$$

It is easy to verify by induction that $\sum_{n=1}^{m-1} H_n^2 = mH_m^2 - (2m+1)H_m + 2m$, therefore

$$\begin{aligned} \sum_{k=1}^{N} \frac{1}{k^2} \sum_{n=1}^{k-1} H_n^2 - \sum_{n=1}^{N} \frac{H_n^2}{n} &= \sum_{k=1}^{N} \frac{1}{k^2} \left(kH_k^2 - (2k+1)H_k + 2k\right) - \sum_{n=1}^{N} \frac{H_n^2}{n} \\ &= -2\sum_{k=1}^{N} \frac{H_k}{k} - \sum_{k=1}^{N} \frac{H_k}{k^2} + 2H_N \\ &= -H_N^2 - \sum_{k=1}^{N} \frac{1}{k^2} - \sum_{k=1}^{N} \frac{H_k}{k^2} + 2H_N \\ &= -H_N^2 + 2H_N - \zeta(2) - 2\zeta(3) + o(1) \end{aligned}$$

where at the last step we used the known results $\sum_{n=1}^{\infty} \frac{1}{k^2} = \zeta(2)$ and $\sum_{k=1}^{\infty} \frac{H_k}{k^2} = 2\zeta(3)$.
On the other hand, since $H_N = \ln(N) + \gamma + o(1)$, and $\sum_{k=1}^{N} \frac{1}{k^2} = \zeta(2) - \frac{1}{N} + O(1/N^2)$, we have that

$$\begin{aligned} \left(\zeta(2) - \sum_{k=1}^{N} \frac{1}{k^2}\right) \cdot \sum_{n=1}^{N} H_n^2 &= \left(\frac{1}{N} + O(1/N^2)\right) \cdot \left((N+1)H_N^2 - (2N+1)H_N + 2N\right) \\ &= H_N^2 - 2H_N + 2 + o(1). \end{aligned}$$

Finally, as $N \to \infty$,

$$S_N = -H_N^2 + 2H_N - \zeta(2) - 2\zeta(3) + H_N^2 - 2H_N + 2 + o(1) \to 2 - \zeta(2) - 2\zeta(3).$$

$\square$