Problem 12101
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Proposed by Hojoo Lee (South Korea).

Find the least upper bound of

$$\sum_{n=1}^{\infty} \frac{\sqrt{x_{n+1}} - \sqrt{x_n}}{\sqrt{(1 + x_{n+1})(1 + x_n)}}$$

over all increasing sequences $x_1, x_2, \ldots$ of positive real numbers.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. We will show that the least upper bound is $\pi/2$ (and it is not attained).
Let $0 < a \leq b$ and let $\alpha = \arctan(\sqrt{a})$ and $\beta = \arctan(\sqrt{b})$ then

$$\frac{\sqrt{b} - \sqrt{a}}{\sqrt{1 + a}\sqrt{1 + b}} = (\tan(\beta) - \tan(\alpha)) \cos(\alpha) \cos(\beta) = \sin(\beta - \alpha) \leq \beta - \alpha = \arctan(\sqrt{b}) - \arctan(\sqrt{a}).$$

Hence, since the sequence $\{x_n\}_{n\geq 1}$ is positive and increasing it follows that

$$\sum_{n=1}^{\infty} \frac{\sqrt{x_{n+1}} - \sqrt{x_n}}{\sqrt{(1 + x_{n+1})(1 + x_n)}} \leq \sum_{n=1}^{\infty} \left(\arctan(\sqrt{x_{n+1}}) - \arctan(\sqrt{x_n})\right)$$
$$= \arctan(\sqrt{L}) - \arctan(\sqrt{x_1}) \leq \frac{\pi}{2} - \arctan(\sqrt{x_1}) < \frac{\pi}{2}$$

where $L = \sup_{n\geq 1}\{x_n\} \in (0, +\infty]$. So $\pi/2$ is a strict upper bound.
Let N be a positive integer and let $x_n = \tan^2\left(\frac{n\pi}{2N}\right)$ for $n = 1, \ldots N - 1$ and $x_n = x_{N-1} + n - (N - 1)$ for all $n \geq N$. Then $\{x_n\}_{n\geq 1}$ is positive and strictly increasing and as $N \to +\infty$

$$\sum_{n=1}^{\infty} \frac{\sqrt{x_{n+1}} - \sqrt{x_n}}{\sqrt{(1 + x_{n+1})(1 + x_n)}} > \sum_{n=1}^{N-2} \frac{\sqrt{x_{n+1}} - \sqrt{x_n}}{\sqrt{(1 + x_{n+1})(1 + x_n)}}$$
$$= \sum_{n=1}^{N-2} \sin\left(\frac{\pi}{2N}\right) = (N - 2) \sin\left(\frac{\pi}{2N}\right) \to \frac{\pi}{2}$$

and we may conclude that $\pi/2$ is the least upper bound. □