Problem 12099
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Proposed by M. Bataille (France).

Let m and n be integers with $0 \leq m \leq n-1$. Evaluate

$$\sum_{k=0,k\neq m}^{n-1} \cot^2\left(\frac{(m-k)\pi}{n}\right).$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Let $j = m-k$ then

$$\begin{aligned}\sum_{k=0,k\neq m}^{n-1} \cot^2\left(\frac{(m-k)\pi}{n}\right) &= \sum_{j=m-n+1}^{-1} \cot^2\left(\frac{j\pi}{n}\right) + \sum_{j=1}^{m} \cot^2\left(\frac{j\pi}{n}\right) \\ &= \sum_{j=m+1}^{n-1} \cot^2\left(\frac{(j-n)\pi}{n}\right) + \sum_{j=1}^{m} \cot^2\left(\frac{j\pi}{n}\right) \\ &= \sum_{j=1}^{n-1} \cot^2\left(\frac{j\pi}{n}\right)\end{aligned}$$

which means that the given sum does not depend on m.
Moreover, by De Moivre's formula,

$$\sin(nt) = \operatorname{Im}\left((\cos(t)+i\sin(t))^n\right) = \sum_{k=0}^{\lfloor n/2\rfloor} (-1)^k \binom{n}{2k+1} \cos^{n-(2k+1)}(t)\sin^{2k+1}(t).$$

Hence, by letting $t = j\pi/n$ with $1 \leq j \leq n-1$, we have that

$$\sum_{k=0}^{\lfloor n/2\rfloor} (-1)^k \binom{n}{2k+1} \cot^{n-(2k+1)}\left(\frac{j\pi}{n}\right) = 0,$$

which implies that the polynomial

$$P(z) := \sum_{k=0}^{\lfloor n/2\rfloor} (-1)^k \binom{n}{2k+1} z^{n-(2k+1)} = nz^{n-1} - \binom{n}{3} z^{n-3} + \dots$$

has degree $n-1$ and it has $n-1$ distinct solutions given by $z_j := \cot\left(\frac{j\pi}{n}\right)$ for $1 \leq j \leq n-1$. Therefore, by Vieta's formulas,

$$\begin{aligned}\sum_{k=0,k\neq m}^{n-1} \cot^2\left(\frac{(m-k)\pi}{n}\right) &= \sum_{j=1}^{n-1} z_j^2 = \left(\sum_{j=1}^{n-1} z_j\right)^2 - 2\left(\sum_{1\leq j<k\leq n-1} z_j z_k\right) \\ &= 0 - 2\,\frac{-\binom{n}{3}}{n} = \frac{(n-1)(n-2)}{3}.\end{aligned}$$

□