Problem 12098
(American Mathematical Monthly, Vol.126, March 2019)

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Suppose that the centroid of a triangle with semiperimeter s and inradius r lies on its incircle. Prove $s \geq 3\sqrt{6}r$, and determine conditions for equality.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Let G be the centroid and I the incenter of the triangle, then

$$36|IG|^2 = 4s^2 + 20r^2 - 64Rr$$

where R is the circumcenter (see p. 51 in *Geometric inequalities* by O. Bottema and others). Since G lies on the incircle $|IG| = r$ and we get $s^2 = 4r^2 + 16Rr$ and the desired inequality is equivalent to $4r^2 + 16Rr \geq 54r^2$ that is

$$8R \geq 25r.$$

By letting $a = y + z$, $b = z + x$, $c = x + y$ where a, b, c are the sides lengths with $x, y, z > 0$, the condition $s^2 = 4r^2 + 16Rr$ can be written as

$$(x + y + z)^2 = \frac{4xyz}{x + y + z} + \frac{4(x + y)(y + z)(z + x)}{x + y + z}$$

that is

$$x^2 + y^2 + z^2 = 2(xy + xz + zy)$$

and by solving it with respect to z we obtain $z = (\sqrt{x} \pm \sqrt{y})^2$.
On the other hand, the inequality $8R \geq 25r$ reduces to

$$\frac{8abc}{4A} \geq \frac{25A}{s}$$

or

$$2(x + y)(y + z)(z + x) - 25xyz \geq 0$$

that is

$$2(x+y)(y+(\sqrt{x}\pm\sqrt{y})^2)((\sqrt{x}\pm\sqrt{y})^2+x)-25xy(\sqrt{x}\pm\sqrt{y}))^2 = (\sqrt{x}\mp\sqrt{y})^2(\sqrt{x}\pm2\sqrt{y})^2(2\sqrt{x}\pm\sqrt{y})^2 \geq 0$$

which holds. Moreover, equality is obtained if and only if $4x = 4y = z$, or $x = 4y = 4z$, or $4x = y = 4z$ that is if and only if the triangle is similar to the isosceles triangle with sides lengths 5, 5, and 2. $\square$