Problem 12094
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Prove

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{16^n (n+1)^3} = 16\log(2) - \frac{32G}{\pi} + \frac{48}{\pi} - 16$$

where $G = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}$ *is the Catalan's constant.*

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Let

$$C_n := \frac{\binom{2n}{n}}{n+1} = \frac{2^{2n+1}}{\pi} B(n+1/2, 3/2) = \frac{2\cdot 4^n}{\pi}\int_0^1 x^n \sqrt{\frac{1-x}{x}}\, dx$$

be the nth Catalan number where $B(s,t)$ denotes the Beta function. For $x \in (0,1)$,

$$\sum_{n=0}^{\infty} \frac{C_n (\frac{x}{4})^n}{n+1} = \frac{1}{x}\int_0^x \sum_{n=0}^{\infty} C_n \left(\frac{t}{4}\right)^n dt = \frac{2}{x}\int_0^x \frac{1-\sqrt{1-t}}{t}\, dt = \frac{4}{x}\left(1-\sqrt{1-x}+\log\left(\frac{1+\sqrt{1-x}}{2}\right)\right).$$

Therefore, by the above integral representation of C_n, after letting $x = \cos^2(t)$ we find

$$\begin{aligned} S :&= \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{16^n (n+1)^3} = \int_0^1 \sqrt{\frac{1-x}{x}} \sum_{n=0}^{\infty} \frac{C_n(\frac{x}{4})^n}{n+1}\, dx \\ &= \frac{8}{\pi}\int_0^1 \frac{\sqrt{1-x}-1+x+\sqrt{1-x}\log(\frac{1+\sqrt{1-x}}{2})}{x^{3/2}}\, dx \\ &= \frac{16}{\pi}\int_0^{\pi/2} \left(\tan^2(t) - \tan^2(t)\sin(t) + \tan^2(t)\log\left(\frac{1+\sin(t)}{2}\right)\right) dt. \end{aligned}$$

Now we have that

$$\int_0^{\pi/2} \left(\tan^2(t) - \tan^2(t)\sin(t)\right) dt = \left[\tan(t)(1-\sin^3(t)) - (2+\sin^2(t))\cos(t) - t\right]_0^{\pi/2} = 2 - \frac{\pi}{2},$$

and

$$\begin{aligned} \int_0^{\pi/2} (1+\tan^2(t))\log\left(\frac{1+\sin(t)}{2}\right) dt &= \int_0^{\pi/2} D(\tan(t))\log\left(\frac{1+\sin(t)}{2}\right) dt \\ &= 0 - \int_0^{\pi/2} \frac{\sin(t)}{1+\sin(t)}\, dt = -\left[t + \frac{2}{1+\tan(t/2)}\right]_0^{\pi/2} = 1 - \frac{\pi}{2}. \end{aligned}$$

Moreover, since

$$\begin{aligned} \log(1+\cos(t)) &= \log(1+e^{it}) + \log(1+e^{-it}) - \log(2) \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}e^{ikx}}{k} + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}e^{-ikx}}{k} - \log(2) = 2\sum_{k=1}^{\infty} \frac{(-1)^{k-1}\cos(kx)}{k} - \log(2) \end{aligned}$$

it follows that

$$\begin{aligned} \int_0^{\pi/2} \log\left(\frac{1+\sin(t)}{2}\right) dt &= \int_0^{\pi/2} \log(1+\cos(t))\, dt - \frac{\pi\log(2)}{2} = 2\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}\int_0^{\pi/2} \cos(kx)\, dx - \pi\log(2) \\ &= 2\sum_{k=1}^{\infty} \frac{(-1)^{k-1}\sin(k\pi/2)}{k^2} - \pi\log(2) = 2G - \pi\log(2). \end{aligned}$$

Finally

$$S = \frac{16}{\pi}\left(\left(2-\frac{\pi}{2}\right) + \left(1-\frac{\pi}{2}\right) - (2G - \pi\log(2))\right) = 16\log(2) - \frac{32G}{\pi} + \frac{48}{\pi} - 16.$$

□