Problem 12092
(American Mathematical Monthly, Vol.126, February 2019)

Proposed by M. Diao and A. Wu (USA).

Let ABC be a triangle, and let P be a point in the plane of the triangle satisfying $\angle BAP = \angle CAP$. Let Q and R be diametrically opposite P on the circumcircles of $\triangle ABP$ and $\triangle ACP$, respectively. Let X be the point of concurrency of line BR and line CQ. Prove that XP and BC are perpendicular.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Without loss of generality we may assume that $A = (0,0)$, $B = b(\cos(\theta), \sin(\theta))$, and $C = c(\cos(\theta), -\sin(\theta)$ with $b > 0$, $c > 0$ and $0 < \theta < \pi$. Let $P = (t, 0)$ with $t \in \mathbb{R}$.
Then it is easy to find

$$Q = \left(0, \frac{-t\cos(\theta) + b}{\sin(\theta)}\right) \quad \text{and} \quad R = \left(0, \frac{t\cos(\theta) - c}{\sin(\theta)}\right).$$

Moreover the point of concurrency of line BR and line CQ is

$$X = \left(\frac{-(b+c)bc\cos(\theta) + 2tbc\cos^2(\theta)}{t(b+c)\cos(\theta) - (b+c)^2 + 2bc\cos^2(\theta)}, \frac{(b-c)\cos(\theta)}{\sin(\theta)} \cdot \frac{(t^2+bc)\cos(\theta) - t(b+c)}{t(b+c)\cos(\theta) - (b+c)^2 + 2bc\cos^2(\theta)}\right)$$

Therefore

$$X - P = \left(-(b+c), \frac{(b-c)\cos(\theta)}{\sin(\theta)}\right) \cdot \frac{(t^2+bc)\cos(\theta) - t(b+c)}{t(b+c)\cos(\theta) - (b+c)^2 + 2bc\cos^2(\theta)}.$$

Finally, in order to show that XP and BC are perpendicular, it suffices to verify that the following dot product is zero:

$$\left(-(b+c), \frac{(b-c)\cos(\theta)}{\sin(\theta)}\right) \cdot ((b-c)\cos(\theta), (b+c)\sin(\theta)) = 0.$$

$\square$