

Problem 12091

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Proposed by Cornel Ioan Vălean (Romania).

Prove

$$2 \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{i!j!k!}{ij(i+j+k)!} (H_{i+j+k} - H_k) = \zeta(3)$$

where H_k is the k -th harmonic number.

Solution proposed by Tewodros Amdeberhan, Department of Mathematics, Tulane University, New Orleans, USA, and Roberto Tauraso, Dipartimento di Matematica, Università di Roma “Tor Vergata”, Roma, Italy.

Solution. Suppressing x , let $F(i, j, k) := \frac{(i-1)!(j-1)!(x)_k}{(x)_{i+j+k}}$. Then, one can *routinely* check that

$$F(i, j, k+1) + F(i, j, k) = F(i, j, k) - F(i+1, j, k) + F(i, j, k) - F(i, j+1, k).$$

Sum this over all positive integers i and j and apply telescoping on the right-hand side so that

$$\sum_{i,j \geq 1} F(i, j, k+1) + \sum_{i,j \geq 1} F(i, j, k) = \sum_{j \geq 1} F(1, j, k) + \sum_{i \geq 1} F(i, 1, k) = 2 \sum_{i \geq 1} F(i, 1, k). \quad (1)$$

Now, let $G_1(i, k) := \frac{(i-1)!(x)_k}{(x)_{i+k}(x+k)}$ then $F(i, 1, k) = G_1(i, k) - G_1(i+1, k)$ and we get

$$\sum_{i \geq 1} F(i, 1, k) = \sum_{i \geq 1} (G_1(i, k) - G_1(i+1, k)) = G_1(1, k) = \frac{1}{(x+k)^2}. \quad (2)$$

Combining (1) and (2), we have

$$\begin{aligned} 2 \sum_{i,j,k \geq 1} F(i, j, k) &= \sum_{i,j \geq 1} F(i, j, 1) + \sum_{i,j,k \geq 1} F(i, j, k+1) + \sum_{i,j,k \geq 1} F(i, j, k) \\ &= f(x) + 2 \sum_{i,k \geq 1} F(i, 1, k) = f(x) + 2 \sum_{k \geq 1} \frac{1}{(x+k)^2} \end{aligned}$$

where

$$f(x) := \sum_{i,j \geq 1} F(i, j, 1) = \sum_{i \geq 1} \sum_{j \geq 1} (G_2(i, j) - G_2(i, j+1)) = \sum_{i \geq 1} G_2(i, 1) = \sum_{i \geq 1} \frac{(i-1)!}{(x+1)_i (x+i)}$$

with $G_2(i, j) := \frac{(i-1)!(j-1)!}{(x+1)_{i+j-1}(x+i)}$ and $F(i, j, 1) = G_2(i, j) - G_2(i, j+1)$. Note that

$$\begin{aligned} f'(1) &= \sum_{i \geq 1} \frac{1}{i(i+1)^2} \left(-H_i + 1 - \frac{2}{i+1} \right) \\ &= \sum_{i \geq 1} \left(\frac{2}{(i+1)^3} + \frac{H_i}{(i+1)^2} + \frac{1}{i+1} - \frac{1}{i} + \frac{1}{(i+1)^2} - \frac{1}{i^2} + \frac{H_i}{i+1} - \frac{H_{i-1}}{i} \right) \\ &= 2(\zeta(3) - 1) + \zeta(3) - 1 - 1 = 3\zeta(3) - 4. \end{aligned}$$

Consequently,

$$\begin{aligned} 2 \sum_{i,j,k \geq 1} \frac{i!j!k! (H_{i+j+k} - H_k)}{ij(i+j+k)!} &= -\frac{d}{dx} \left(2 \sum_{i,j,k \geq 1} F(i, j, k) \right) \Big|_{x=1} \\ &= -f'(1) + 4 \sum_{k \geq 1} \frac{1}{(1+k)^3} \\ &= -3\zeta(3) + 4 + 4(\zeta(3) - 1) = \zeta(3). \end{aligned}$$

□