Problem 12090
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The Pell-Lucas numbers Q_n *satisfy* $Q_0 = 2$, $Q_1 = 2$, *and* $Q_n = 2Q_{n-1} + Q_{n-2}$ *for* $n \geq 2$. *Prove*

$$\sum_{n=1}^{\infty} \arctan\left(\frac{2}{Q_n}\right) \arctan\left(\frac{2}{Q_{n+1}}\right) = \frac{\pi^2}{32}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. We first note that $Q_n = q^n + (-q)^{-n}$ with $q = 1 + \sqrt{2}$.
Therefore, if n is even

$$\arctan\left(\frac{2}{Q_n}\right) = \arctan\left(\frac{2}{q^n + q^{-n}}\right) = \arctan\left(\frac{q^{-(n-1)} - q^{-(n+1)}}{1 + q^{-(n-1)} \cdot q^{-(n+1)}}\right)$$
$$= \arctan(q^{-(n-1)}) - \arctan(q^{-(n+1)}).$$

On the other hand, if n is odd

$$\arctan\left(\frac{2}{Q_n}\right) = \arctan\left(\frac{2}{q^n - q^{-n}}\right) = \arctan\left(\frac{2q^{-n}}{1 - q^{-2n}}\right) = 2\arctan(q^{-n}).$$

Hence the desired sum is

$$\begin{aligned}
S &= \sum_{n=1}^{\infty} \arctan\left(\frac{2}{Q_{2n-1}}\right) \arctan\left(\frac{2}{Q_{2n}}\right) + \sum_{n=1}^{\infty} \arctan\left(\frac{2}{Q_{2n}}\right) \arctan\left(\frac{2}{Q_{2n+1}}\right) \\
&= \sum_{n=1}^{\infty} \arctan\left(\frac{2}{Q_{2n}}\right) \left(\arctan\left(\frac{2}{Q_{2n-1}}\right) + \arctan\left(\frac{2}{Q_{2n+1}}\right)\right) \\
&= 2\sum_{n=1}^{\infty} \left(\arctan(q^{-(2n-1)}) - \arctan(q^{-(2n+1)})\right) \cdot \left(\arctan(q^{-(2n-1)}) + \arctan(q^{-(2n+1)})\right) \\
&= 2\sum_{n=1}^{\infty} \left(\arctan^2(q^{-(2n-1)}) - \arctan^2(q^{-(2n+1)})\right) \\
&= 2\arctan^2(q^{-1}) - 2\lim_{n\to\infty} \arctan^2(q^{-(2n+1)}) \\
&= 2\arctan^2(\sqrt{2} - 1) = 2\left(\frac{\pi}{8}\right)^2 = \frac{\pi^2}{32}.
\end{aligned}$$

□