Problem 12088
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Proposed by F. Stanescu (Romania).

Let k be a positive integer with $k \geq 2$, and let $f : [0,1] \to \mathbb{R}$ be a function with continuous k-th derivative. Suppose $f^{(k)}(x) \geq 0$ for all $x \in [0,1]$, and suppose $f^{(i)}(0) = 0$ for all $i \in \{0,1,\ldots,k-2\}$. Prove

$$\int_0^1 x^{k-1} f(1-x)\,dx \leq \frac{(k-1)!k!}{(2k-1)!}\int_0^1 f(x)\,dx.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Since $f^{(i)}(0) = 0$ for $i = 0,1,\ldots,k-2$, by integration by parts,

$$\begin{aligned}\int_0^1 x^{k-1} f(1-x)\,dx &= \left[\frac{x^k}{k}\,f(1-x)\right]_0^1 + \int_0^1 \frac{x^k}{k}\,f^{(1)}(1-x)\,dx = \int_0^1 \frac{x^k}{k}\,f^{(1)}(1-x)\,dx\\ &= \left[\frac{x^{k+1}}{k(k+1)}\,f^{(1)}(1-x)\right]_0^1 + \int_0^1 \frac{x^{k+1}}{k(k+1)}\,f^{(2)}(1-x)\,dx = \int_0^1 \frac{x^{k+1}}{k(k+1)}\,f^{(2)}(1-x)\,dx\\ &= \left[\frac{x^{2k-1}}{k(k+1)\cdots(2k-1)}\,f^{(k-1)}(1-x)\right]_0^1 + \int_0^1 \frac{x^{2k-1}}{k(k+1)\cdots(2k-1)}\,f^{(k)}(1-x)\,dx\\ &= \frac{(k-1)!}{(2k-1)!}\left(f^{(k-1)}(0) + \int_0^1 x^{2k-1}\,f^{(k)}(1-x)\,dx\right).\end{aligned}$$

On the other hand, by the integral form of the remainder in Taylor's Theorem, we have that for $x \in [0,1]$,

$$\begin{aligned}f(x) &= f(0) + f'(0)x + \cdots + \frac{f^{(k-1)}(0)}{(k-1)!}x^{k-1} + \int_0^x \frac{(x-t)^{k-1}}{(k-1)!}\,f^{(k)}(t)\,dt\\ &= \frac{f^{(k-1)}(0)}{(k-1)!}x^{k-1} + \int_0^x \frac{(x-t)^{k-1}}{(k-1)!}\,f^{(k)}(t)\,dt,\end{aligned}$$

which implies

$$\begin{aligned}\int_0^1 f(x)\,dx &= \frac{f^{(k-1)}(0)}{(k-1)!}\int_0^1 x^{k-1}\,dx + \int_0^1\left(\int_0^x \frac{(x-t)^{k-1}}{(k-1)!}\,f^{(k)}(t)\,dt\right)dx\\ &= \frac{f^{(k-1)}(0)}{k!} + \int_0^1 f^{(k)}(t)\left(\int_t^1 \frac{(x-t)^{k-1}}{(k-1)!}\,dx\right)dt\\ &= \frac{1}{k!}\left(f^{(k-1)}(0) + \int_0^1 (1-t)^k f^{(k)}(t)\,dt\right) = \frac{1}{k!}\left(f^{(k-1)}(0) + \int_0^1 x^k f^{(k)}(1-x)\,dx\right).\end{aligned}$$

Hence, the given inequality is equivalent to

$$\frac{(k-1)!}{(2k-1)!}\left(f^{(k-1)}(0) + \int_0^1 x^{2k-1}\,f^{(k)}(1-x)\,dx\right) \leq \frac{(k-1)!}{(2k-1)!}\left(f^{(k-1)}(0) + \int_0^1 x^k f^{(k)}(1-x)\,dx\right)$$

that is

$$\int_0^1 x^{2k-1}\,f^{(k)}(1-x)\,dx \leq \int_0^1 x^k f^{(k)}(1-x)\,dx$$

which holds because $f^{(k)}(x) \geq 0$ and $x^{2k-1} \leq x^k$ for $x \in [0,1]$. □