**Problem 12086**
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*Let $ABC$ be a triangle with right angle at $A$, and let $H$ be the foot of the altitude from $A$. Let $M$, $N$, and $P$ be the incenters of triangles $ABH$, $ABC$, and $ACH$, respectively. Prove that the ratio of the area of triangle $MNP$ to the area of triangle $ABC$ is at most $(\sqrt{2}-1)^3/2$, and determine when equality holds.*

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

*Solution.* Without loss of generality, we may assume that $B=(-1,0)$, $C=(1,0)$ and $A=(\cos(\theta),\sin(\theta))$ with $\theta\in[0,\pi]$. Then $H=(\cos(\theta),0)$ and

$$M=(\cos(\theta)-r_B,r_B),\quad N=(\cos(\theta)-(\sin(\theta)-r)\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right),r),\quad P=(\cos(\theta)+r_C,r_C)$$

where $r_B$, $r$ and $r_C$ are the inradii of the triangles $ABH$, $ABC$, and $ACH$ respectively,

$$r_B=\frac{\sin(\theta)\cos^2(\frac{\theta}{2})}{\cos(\frac{\theta}{2})+\cos^2(\frac{\theta}{2})+\frac{1}{2}\sin(\theta)},\ r=\frac{\sin(\theta)}{\sin(\frac{\theta}{2})+\cos(\frac{\theta}{2})+1},\ r_C=\frac{\sin(\theta)\sin^2(\frac{\theta}{2})}{\sin(\frac{\theta}{2})+\sin^2(\frac{\theta}{2})+\frac{1}{2}\sin(\theta)}.$$

Hence

$$\frac{\text{Area}(MNP)}{\text{Area}(ABC)}=\frac{|MN\times PN|}{2\sin(\theta)}=\frac{(1-\sin(\frac{\theta}{2}))(1-\cos(\frac{\theta}{2}))}{\sin(\frac{\theta}{2})+\cos(\frac{\theta}{2})+1}=f(t)\le f(\sqrt{2}+1)=\frac{(\sqrt{2}-1)^3}{2}$$

where $t=\sin(\frac{\theta}{2})+\cos(\frac{\theta}{2})+1=\sqrt{2}\sin\left(\frac{\pi}{4}+\frac{\theta}{2}\right)+1\in[2,\sqrt{2}+1]$ and

$$f(t):=\frac{(t-2)^2}{2t}=\frac{1}{2}\left(t+\frac{4}{t}\right)-2$$

is strictly increasing in $[2,\sqrt{2}+1]$. It follows that equality holds if and only if $t=\sqrt{2}+1$, i.e. $\theta=\frac{\pi}{2}$ or $ABC$ is an isosceles right triangle. □