Problem 12085
(American Mathematical Monthly, Vol.126, January 2019)

Proposed by J. DeVincentis, S. Wagon, and M. Elgersma (USA).

For which positive integers n can $\{1,\ldots,n\}$ be partitioned into two sets A and B of the same size so that

$$\sum_{k\in A} k = \sum_{k\in B} k, \quad \sum_{k\in A} k^2 = \sum_{k\in B} k^2, \quad \text{and} \quad \sum_{k\in A} k^3 = \sum_{k\in B} k^3\,?$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. We show that such partition exists if and only if n is a multiple of 8 with $n \geq 16$.

Let n be a positive integer such that the partition exists. Since $|A| = |B|$, then n has to be even. Hence $n+1$ is odd and $\sum_{k\in A} k = \frac{1}{2}\sum_{k=1}^{n} k = \frac{n(n+1)}{4} \in \mathbb{N}$ implies that 4 divides n. Let $n = 4N$ then, since $k \equiv k^3 \pmod 2$,

$$\begin{cases} \sum_{k\in A} k = \frac{1}{2}\sum_{k=1}^{n} k = \frac{n(n+1)}{4} = N(4N+1) \equiv N \pmod 2 \\ \sum_{k\in A} k^3 = \frac{1}{2}\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{8} = 2N^2(4N+1)^2 \equiv 0 \pmod 2 \end{cases}$$

which implies that $N \equiv 0 \pmod 2$ and therefore n has to be a multiple of 8. It can be verified that there is no such partition for $n = 8$. It remains to show that the partition exists when n is multiple of 8 and $n \geq 16$.

For any non-negative integer x, we have a partition of $\{1+x,\ldots,8+x\}$,

$$A_8(x) := \{1+x, 4+x, 6+x, 7+x)\} \text{ and } B_8(x) := \{2+x, 3+x, 5+x, 8+x\}$$

and a partition of $\{1+x,\ldots,12+x\}$

$$A_{12}(x) := \{1+x, 3+x, 7+x, 8+x, 9+x, 11+x)\} \text{ and } B_{12}(x) := \{2+x, 4+x, 5+x, 6+x, 10+x, 12+x\}$$

such that $|A_8(x)| = |B_8(x)|$, $|A_{12}(x)| = |B_{12}(x)|$ and for $j = 1, 2$,

$$\sum_{k\in A_8(x)} k^j = \sum_{k\in B_8(x)} k^j \quad \text{and} \quad \sum_{k\in A_{12}(x)} k^j = \sum_{k\in B_{12}(x)} k^j.$$

Therefore, if m is positive integer of the form $8a + 12b = 4(2a+3b)$ with $a, b \geq 0$, that is if m is a multiple of 4 with $m \geq 8$, then we have a partition of $\{1,\ldots,m\}$,

$$A' := \bigcup_{j=0}^{a-1} A_8(8j) \cup \bigcup_{j=0}^{b-1} A_{12}(8a+12j) \quad \text{and} \quad B' := \bigcup_{j=0}^{a-1} B_8(8j) \cup \bigcup_{j=0}^{b-1} B_{12}(8a+12j)$$

such that $|A'| = |B'|$ and

$$\sum_{k\in A'} k = \sum_{k\in B'} k, \quad \text{and} \quad \sum_{k\in A'} k^2 = \sum_{k\in B'} k^2.$$

Finally we define a partition of $\{1,\ldots,n\}$ with $n = 2m$,

$$A := A' \cup (m + B') \quad \text{and} \quad B := B' \cup (m + A').$$

Then it is easy to verify that $|A| = |B|$, $\sum_{k\in A} k = \sum_{k\in B} k$, $\sum_{k\in A} k^2 = \sum_{k\in B} k^2$ and

$$\begin{aligned} \sum_{k\in A} k^3 &= \sum_{k\in A'} k^3 + \sum_{k\in B'} (m+k)^3 = \sum_{k\in A'} k^3 + m^3|B'| + 3m^2 \sum_{k\in B'} k + 3m \sum_{k\in B'} k^2 + \sum_{k\in B'} k^3 \\ &= \sum_{k\in B'} k^3 + m^3|A'| + 3m^2 \sum_{k\in A'} k + 3m \sum_{k\in A'} k^2 + \sum_{k\in A'} k^3 = \sum_{k\in B} k^3 \end{aligned}$$

and we are done. □