Problem 12084
(American Mathematical Monthly, Vol.126, January 2019)

Proposed by G. Stoica (Canada).

Let $\{a_n\}_{n\geq 1}$ *be a sequence of nonnegative numbers. Prove that* $\{\frac{1}{n}\sum_{k=1}^n a_k\}_{n\geq 1}$ *is unbounded if and only if there exists a decreasing sequence* $\{b_n\}_{n\geq 1}$ *such that* $\lim_{n\to\infty} b_n = 0$, $\sum_{n=1}^\infty b_n$ *is finite, and* $\sum_{n=1}^\infty a_n b_n$ *is infinite. Is the word "decreasing" essential?*

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Let $A_n = \sum_{k=1}^n a_k$, then

$$\sum_{n=1}^{N} a_n b_n = \sum_{n=1}^{N} A_n b_n - \sum_{n=1}^{N} A_{n-1} b_n = \sum_{n=1}^{N-1} A_n \cdot (b_n - b_{n+1}) + A_N b_N. \tag{1}$$

Proof of $\Leftarrow$. Assume that $\{\frac{1}{n}\sum_{k=1}^n a_k\}_{n\geq 1}$ is bounded, i.e. $0 \leq A_n \leq Mn$ for some $M \in \mathbb{R}$. Let $\{b_n\}_{n\geq 1}$ be any decreasing nonnegative sequence, then by (1),

$$\sum_{n=1}^{N} a_n b_n \leq M \left(\sum_{n=1}^{N-1} n(b_n - b_{n+1}) + N b_N\right) = M \sum_{n=1}^{N} b_n$$

So if $\sum_{n=1}^\infty b_n$ is finite then $\sum_{n=1}^\infty a_n b_n$ is finite too and we have a contradiction.

Proof of $\Rightarrow$. Assume that $\{\frac{1}{n}\sum_{k=1}^n a_k\}_{n\geq 1}$ is unbounded. Then there is a strictly increasing sequence $\{n_k\}_{k\geq 1}$ of positive integers such that $A_{n_k} \geq k n_k$. Let

$$b_i := \sum_{k=j}^{\infty} \frac{1}{n_k k^2} \qquad \text{for } i = n_{j-1}+1, \dots, n_j \text{ with } j \geq 1$$

where $n_0 = 0$. Then $\{b_n\}_{n\geq 1}$ is a decreasing nonnegative sequence such that $\sum_{n=1}^\infty b_n$ is finite:

$$\sum_{n=1}^{\infty} b_n = \sum_{j=1}^{\infty} (n_j - n_{j-1}) \sum_{k=j}^{\infty} \frac{1}{n_k k^2} = \sum_{k=1}^{\infty} \frac{1}{n_k k^2} \sum_{j=1}^{k} (n_j - n_{j-1}) = \sum_{i=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6},$$

and, by (1), $\sum_{n=1}^\infty a_n b_n$ is infinite

$$\sum_{n=1}^{\infty} a_n b_n \geq \sum_{n=1}^{\infty} A_n \cdot (b_n - b_{n+1}) \geq \sum_{k=1}^{\infty} k n_k \cdot \frac{1}{n_k k^2} = \sum_{k=1}^{\infty} \frac{1}{k} = +\infty.$$

Yes, the word "decreasing" is essential: let

$$a_n = \begin{cases} k & \text{if } n = 2^k \text{ with } k \in \mathbb{N}, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad b_n = \begin{cases} \frac{1}{k^2} & \text{if } n = 2^k \text{ with } k \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

then $\{b_n\}_{n\geq 1}$ is a nonnegative sequence (which is not decreasing) such that $\lim_{n\to\infty} b_n = 0$, $\sum_{n=1}^\infty b_n = \sum_{k=1}^\infty \frac{1}{k^2}$ is finite, and $\sum_{n=1}^\infty a_n b_n = \sum_{k=1}^\infty \frac{1}{k}$ is infinite, but $\{\frac{1}{n}\sum_{k=1}^n a_k\}_{n\geq 1}$ is bounded

$$0 \leq \frac{1}{n}\sum_{k=1}^{n} a_k = \frac{1}{n} \sum_{k=1}^{\lfloor \log_2(n) \rfloor} k \leq \frac{\log_2(n)(\log_2(n)+1)}{2n} < 1.$$

□