Problem 12083
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Let x, y, and z be positive real numbers. Prove

$$\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x}\geq\frac{3\sqrt{3}}{2\sqrt{x^2+y^2+z^2}}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. By the Cauchy-Schwarz inequality,

$$((x+y)+(y+z)+(z+x))\cdot\left(\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x}\right)\geq(1+1+1)^2$$

that is

$$\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x}\geq\frac{9}{2(x+y+z)}.$$

So, it remains to show that

$$\frac{9}{2(x+y+z)}\geq\frac{3\sqrt{3}}{2\sqrt{x^2+y^2+z^2}}$$

that is

$$\sqrt{1^2+1^2+1^2}\cdot\sqrt{x^2+y^2+z^2}\geq(x+y+z)$$

which holds again by the Cauchy-Schwarz inequality. $\square$