Problem 12079
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Proposed by M. Omarjee (France).

Choose x_1 *in* $(0,1)$, *and let for* $n \geq 1$,

$$x_{n+1} = \frac{1}{n}\sum_{k=1}^{n} \ln(1+x_k).$$

Compute $\lim_{n\to\infty} x_n \ln(n)$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. We first note that for $n \geq 1$,

$$0 < x_{n+1} = \frac{\log(1+x_n)}{n} + \frac{(n-1)x_n}{n} < \frac{x_n}{n} + \frac{(n-1)x_n}{n} = x_n \leq x_1,$$

which means that the sequence is positive and strictly decreasing. Therefore it has a limit $L \in [0,1)$. Moreover

$$n(x_n - x_{n+1}) = x_n - \log(1+x_n)$$

and since for $t > 0$,

$$\frac{t^2}{2} - \frac{t^3}{3} < t - \log(1+t) < \frac{t^2}{2},$$

we have that

$$\frac{1}{6} = \frac{1}{2} - \frac{1}{3} < \frac{1}{2} - \frac{x_n}{3} < \frac{n(x_n - x_{n+1})}{x_n^2} < \frac{1}{2}.$$

Then

$$\frac{x_n^2}{6n} < x_n - x_{n+1} \implies \sum_{n=1}^{N} \frac{x_n^2}{n} < 6\sum_{n=1}^{N}(x_n - x_{n+1}) = 6(x_1 - x_{N+1}) < 6$$

which implies that $L = 0$ otherwise we have a contradiction: the bounded partial sum on the left diverges as $n \to +\infty$. Therefore, by the above inequality, $x_n \to 0$ implies

$$\lim_{n\to\infty} \frac{n(x_n - x_{n+1})}{x_n^2} = \frac{1}{2}.$$

Finally, by StolzCesaro theorem, we find the desired limit

$$\begin{aligned}\lim_{n\to\infty} x_n \ln(n) &= \lim_{n\to\infty} \frac{\ln(n)}{1/x_n} \stackrel{\text{SC}}{=} \lim_{n\to\infty} \frac{\ln(n+1) - \ln(n)}{1/x_{n+1} - 1/x_n} \\ &= \lim_{n\to\infty} \Big(\underbrace{\frac{\ln(1+1/n) - 1}{1/n}}_{\to 1} \cdot \underbrace{\frac{x_n^2}{n(x_n - x_{n+1})}}_{\to 2} \cdot \Big(\underbrace{\frac{\log(1+x_n)}{nx_n}}_{\to 0} + \underbrace{\frac{n-1}{n}}_{\to 1} \Big)\Big) = 2.\end{aligned}$$

□