Problem 12078
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Proposed by H. Ohtsuka (Japan).

Let $\binom{n}{k}_q$ *be the q-binomial coefficient defined by*

$$\binom{n}{k}_q = \prod_{i=0}^{k-1} \frac{1-q^{n-i}}{1-q^{i+1}}.$$

For a positive integer s *and for* $0<q<1$, *prove*

$$\sum_{n=1}^{\infty} \frac{q^{sn}}{\binom{s+n}{s+1}_q} = \frac{q^s(1-q^{s+1})}{1-q^s}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. We have that

$$\begin{aligned}\frac{q^{s(n-1)}}{\binom{s+n-1}{s}_q} - \frac{q^{sn}}{\binom{s+n}{s}_q} &= \frac{q^{s(n-1)}}{\binom{s+n}{s+1}_q} \cdot \frac{1-q^{s+n}}{1-q^{s+1}} - \frac{q^{sn}}{\binom{s+n}{s+1}_q} \cdot \frac{1-q^n}{1-q^{s+1}} \\ &= \frac{1}{(1-q^{s+1})\binom{s+n}{s+1}_q}\left(q^{s(n-1)}(1-q^{s+n}) - q^{sn}(1-q^n)\right) \\ &= \frac{q^{s(n-1)}-q^{sn}}{(1-q^{s+1})\binom{s+n}{s+1}_q} = \frac{q^{sn}(1-q^s)}{q^s(1-q^{s+1})\binom{s+n}{s+1}_q}.\end{aligned}$$

Hence, for any positive integer N, the partial sum is telescopic,

$$\sum_{n=1}^{N} \frac{q^{sn}}{\binom{s+n}{s+1}_q} = \frac{q^s(1-q^{s+1})}{1-q^s} \sum_{n=1}^{N}\left(\frac{q^{s(n-1)}}{\binom{s+n-1}{s}_q} - \frac{q^{sn}}{\binom{s+n}{s}_q}\right) = \frac{q^s(1-q^{s+1})}{1-q^s}\left(1 - \frac{q^{sN}}{\binom{s+N}{s}_q}\right).$$

It follows that

$$\sum_{n=1}^{\infty} \frac{q^{sn}}{\binom{s+n}{s+1}_q} = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{q^{sn}}{\binom{s+n}{s+1}_q} = \frac{q^s(1-q^{s+1})}{1-q^s}$$

because for $0<q<1$,

$$\lim_{N\to\infty} \frac{q^{sN}}{\binom{s+N}{s}_q} = \lim_{N\to\infty} q^{sN} \frac{\prod_{i=1}^{s}(1-q^i)}{\prod_{i=1}^{s}(1-q^{N+i})} = 0.$$

□