Problem 12077
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Let $f(x)$ be a monic polynomial of degree n with distinct zeros $a_1, \dots, a_n$. Prove

$$\sum_{k=1}^{n} \frac{a_k^{n-1}}{f'(a_k)} = 1.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. We will show the following more general result:

$$\sum_{k=1}^{n} \frac{a_k^{m}}{f'(a_k)} = \begin{cases} 0 & \text{if } 0 \leq m < n-1 \\ 1 & \text{if } m = n-1, \\ \sum_{j=1}^{n} a_j & \text{if } m = n. \end{cases}$$

We first consider the polynomial

$$F(x) := x^{n-1} - \sum_{k=1}^{n} \frac{a_k^{n-1}}{f'(a_k)} \cdot \frac{f(x)}{x - a_k}.$$

Note that the degree of F is $n-1$ and that for $1 \leq j \leq n$,

$$\left. \frac{f(x)}{x - a_k} \right|_{x=a_j} = \begin{cases} 0 & \text{if } j \neq k, \\ f'(a_k) & \text{if } j = k, \end{cases}$$

which implies that $F(a_k) = 0$ for $k = 1, 2, \dots, n$. Since F has n different roots, we may conclude that F is identically zero. Therefore

$$x^{n-1} = \sum_{k=1}^{n} \frac{a_k^{n-1}}{f'(a_k)} \cdot \frac{f(x)}{x - a_k}.$$

For $m = n-1$,

$$1 = [x^{n-1}]x^{n-1} = \sum_{k=1}^{n} \frac{a_k^{n-1}}{f'(a_k)} \cdot [x^{n-1}]\frac{f(x)}{x - a_k} = \sum_{k=1}^{n} \frac{a_k^{n-1}}{f'(a_k)}.$$

In similar way, we show the case $m = n$,

$$\begin{aligned} 0 = [x^{n-2}]x^{n-1} &= \sum_{k=1}^{n} \frac{a_k^{n-1}}{f'(a_k)} \cdot [x^{n-2}]\frac{f(x)}{x - a_k} \\ &= \sum_{k=1}^{n} \frac{a_k^{n-1}}{f'(a_k)} \cdot \left(a_k - \sum_{j=1}^{n} a_j \right) = \sum_{k=1}^{n} \frac{a_k^{n}}{f'(a_k)} - \sum_{j=1}^{n} a_j. \end{aligned}$$

Finally for $0 \leq m < n$, let $R > \max_{1 \leq j \leq n} |a_j|$. Then, by Cauchy's theorem,

$$\frac{1}{2\pi i} \oint_{|z|=R} \frac{z^m}{f(z)} \, dz = \sum_{k=1}^{n} \operatorname{Res}\left(\frac{z^m}{f(z)}; a_k \right) = \sum_{k=1}^{n} \frac{a_k^{m}}{f'(a_k)}$$

where we used the fact that $a_1, \dots, a_n$ are simple poles. Since $m < n-1$, the limit of the left-hand side as $R \to +\infty$ is zero and we are done. □