Problem 12076
(American Mathematical Monthly, Vol.125, December 2018)

Proposed by T. Beke (USA).

From each of the three feet of the altitudes of an arbitrary triangle, produce two points by projecting this foot onto the other two sides. Show that the six points produced in this way are concyclic.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Let A_{AB} an A_{CA} be the points obtained by projecting the foot of the altitude H_A dropped from vertex A, respectively onto the sides AB and CA. The points B_{AB}, B_{BC}, C_{CA}, and C_{BC} are defined in a similar way. Note that

$$|AA_{AB}| = |AH_A|\sin(B) = |AB|\sin^2(B) \quad \text{and} \quad |AA_{CA}| = |AH_A|\sin(C) = |CA|\sin^2(C).$$

In the same way we obtain

$$|BB_{AB}| = |AB|\sin^2(A),\ |BB_{BC}| = |BC|\sin^2(C),\ |CC_{CA}| = |CA|\sin^2(A),\ |CC_{BC}| = |BC|\sin^2(B).$$

Therefore, by the law of sines $|AB|\sin(B) = |CA|\sin(C)$,

$$|AB_{AB}| \cdot |AA_{AB}| = |AB|^2\cos^2(A)\sin^2(B) = |CA|^2\cos^2(A)\sin^2(C) = |AC_{CA}| \cdot |AA_{CA}|$$

which implies, by the power of a point theorem, that $B_{AB}, A_{AB}, C_{CA}, A_{CA}$ are concyclic. In the same way we may conclude that $B_{AB}, A_{AB}, C_{BC}, B_{BC}$ are concyclic and $C_{BC}, B_{BC}, C_{CA}, A_{CA}$ are concyclic.

The circle through B_{AB},A_{AB},C_{CA},A_{CA} and the circle through B_{AB},A_{AB},C_{BC},B_{BC} have AB as the radical axis. Similarly, BC is the radical axis of the circle through B_{AB},A_{AB}, C_{BC},B_{BC} and the circle through C_{BC},B_{BC},C_{CA},A_{CA}, while CA is the radical axis of the circle through C_{BC},B_{BC},C_{CA},A_{CA} and the circle through B_{AB},A_{AB},C_{CA},A_{CA}. But the pairwise radical axes of any three circles are known to intersect at their radical center. Since they are the sides of a triangle we have a contradiction and the six points B_{AB},A_{AB},C_{BC},B_{BC},A_{CA},C_{CA} are concyclic. □