Problem 12073
(American Mathematical Monthly, Vol.125, November 2018)

Proposed by H. Karakus (Turkey).

Given a scalene triangle ABC, let G denote its centroid and H denote its orthocenter. Let P_A be the second point of intersection of the two circles through A that are tangent to BC at B and at C. Similarly define P_B and P_C. Prove that G, H, P_A, P_B, and P_C are concyclic.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. We prove that P_A, P_B, and P_C belong to the circle with diameter the segment GH. It suffices to show that the line HP_A is orthogonal to the line GP_A. The property holds for any non-degenerate triangle.

Without loss of generality we may assume that A, B, C are complex numbers along the unit circle $|z| = 1$. Then

$$G = \frac{A+B+C}{3} \quad \text{and} \quad H = A+B+C.$$

Moreover, the intersection X between the line AP_A and the common tangent of the two circles through A is the midpont of the side BC. Indeed, by considering the powers of such point X with respect to the two circles, we have that

$$|X-B|^2 = |X-P_A| \cdot |X-A| = |X-C|^2 \implies X = M = \frac{B+C}{2}$$

which implies

$$P_A = M - t(M-A) \quad \text{where } t = \frac{|M-P_A|}{|M-A|} = \frac{|M-C|^2}{|M-A|^2} = \frac{|B-C|^2}{4|M-A|^2}.$$

Hence, since both G and P_A are along the line AM, it remains to show that the line HP_A is orthogonal to line AM, that is the following scalar product is zero:

$$\begin{aligned}
\operatorname{Re}\left((H-P_A)\cdot\overline{(M-A)}\right) &= \operatorname{Re}\left((M+A+t(M-A))\cdot\overline{(M-A)}\right)\\
&= \operatorname{Re}\left((M+A)\cdot\overline{(M-A)}\right) + t|M-A|^2\\
&= |M|^2 - 1 + \frac{|B-C|^2}{4}\\
&= \frac{|B+C|^2}{4} - 1 + \frac{|B-C|^2}{4} = 0.
\end{aligned}$$

□