Problem 12067
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Proposed by P. Bracken (USA).

For a positive integer n, let

$$\beta_n = 6n + 12n^2(\gamma - \gamma_n)$$

where $\gamma_n = H_n - \ln(n)$, $H_n = \sum_{j=1}^{n} \frac{1}{j}$ is the n-th harmonic number, and γ is the Euler's constant. Prove that $\beta_{n+1} > \beta_n$ for all n.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. It is known that the digamma function ψ has the following integral representation: if the real part of z is positive then

$$\psi(z) = \ln(z) - \frac{1}{2z} - 2\int_0^\infty \frac{x}{(z^2 + x^2)(e^{2\pi x} - 1)}\, dx$$

(for example see 6.3.21 in *Handbook of Mathematical Functions* by Abramowitz and Stegun). Hence, for a positive integer n, since $\psi(n) = H_{n-1} - \gamma$, we have that

$$\begin{aligned}\beta_n &= 6n + 12n^2(\gamma - \gamma_n) = 6n + 12n^2\left(\ln(n) - \frac{1}{n} - \psi(n)\right)\\ &= 12n^2\left(\ln(n) - \frac{1}{2n} - \psi(n)\right) = 24\int_0^\infty \frac{x}{(1 + \frac{x^2}{n^2})(e^{2\pi x} - 1)}\, dx.\end{aligned}$$

Finally for $x > 0$,

$$\frac{x}{(1 + \frac{x^2}{(n+1)^2})(e^{2\pi x} - 1)} > \frac{x}{(1 + \frac{x^2}{n^2})(e^{2\pi x} - 1)}$$

which implies $\beta_{n+1} > \beta_n$. □

Remark. By using the known asymptotic expansion

$$H_n \sim \ln n + \gamma + \frac{1}{2n} - \sum_{k=1}^{\infty} \frac{B_{2k}}{2kn^{2k}} = \ln n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} + o\left(\frac{1}{n^4}\right).$$

we get that, as n goes to infinity,

$$\beta_n = 6n + 12n^2(\ln(n) + \gamma - H_n) = 1 - \frac{1}{10n^2} + o\left(\frac{1}{n^2}\right) \to 1.$$