Problem 12066
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Proposed by Xiang-Qian Chang (USA).

Let n and k be integers greater than 1, and let A be an $n \times n$ positive definite Hermitian matrix. Prove

$$(\det(A))^{1/n} \leq \left(\frac{\operatorname{trace}^k(A) - \operatorname{trace}(A^k)}{n^k - n}\right)^{1/k}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. The matrix A is diagonalizable and all its eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ are real and positive numbers. Hence the given inequality is equivalent to

$$\left(\prod_{j=1}^{n} \lambda_j^k\right)^{1/n} \leq \frac{\left(\sum_{j=1}^{n} \lambda_j\right)^k - \left(\sum_{j=1}^{n} \lambda_j^k\right)}{n^k - n}. \qquad (\star)$$

Now the right-hand side of $(\star)$ is an arithmetic mean:

$$\frac{\left(\sum_{j=1}^{n} \lambda_j\right)^k - \left(\sum_{j=1}^{n} \lambda_j^k\right)}{n^k - n} = \frac{1}{|S|} \sum_{\mathbf{j} \in S} \lambda_{j_1} \lambda_{j_2} \cdots \lambda_{j_k}$$

where S is the set of k-ple $\mathbf{j} = (j_1, j_2, \ldots, j_k) \in \{1, 2, \ldots, n\}^k$ such that $j_a \neq j_b$ for some $a \neq b$.
By the AM-GM inequality, this arithmetic mean is greater than or equal to the corresponding geometric mean. Since each index j appears as the i-th component of $\mathbf{j} \in S$ exactly $n^{k-1} - 1$ times, we have that such geometric mean can be written as

$$\left(\prod_{\mathbf{j} \in S} \lambda_{j_1} \lambda_{j_2} \cdots \lambda_{j_k}\right)^{1/|S|} = \left(\prod_{j=1}^{n} (\lambda_j^{n^{k-1}-1})^k\right)^{1/(n^k - n)} = \left(\prod_{j=1}^{n} \lambda_j^k\right)^{1/n}$$

which is just the left-hand side of $(\star)$. □