Problem 12064
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Let f be a convex, continuously differentiable function from $[1, \infty)$ *to* $\mathbb{R}$ *such that* $f'(x) > 0$ *for all* $x \geq 1$*. Prove that the improper integral*

$$\int_1^\infty \frac{dx}{f'(x)}$$

is convergent if and only if the series

$$\sum_{n=1}^{\infty}(f^{-1}(f(n)+\epsilon)-n)$$

is convergent for all $\epsilon > 0$*.*

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Since f is convex with positive derivative, it follows that f is strictly increasing, f' is increasing and

$$f(x) \geq f'(1)(x-1) + f(1)$$

which implies that $\lim_{x\to+\infty} f(x) = +\infty$. Hence $f([1, +\infty)) = [f(1), +\infty)$, and the inverse d f^{-1} is a continuous, differentiable, and strictly increasing function in $[f(1), +\infty)$.

Given $\epsilon > 0$, for any positive integer n, we define $x_n = f^{-1}(f(n) + \epsilon) > n$. By the Mean Value Theorem applied to f^{-1}, there is $s_n \in (f(n), f(x_n))$ such that

$$f^{-1}(f(n)+\epsilon) - n = f^{-1}(f(x_n)) - f^{-1}(f(n)) = D(f^{-1})(s_n)\cdot(f(x_n) - f(n)) = \frac{\epsilon}{f'(t_n)}$$

where $t_n = f^{-1}(s_n) \in (n, x_n)$.

Hence $f'(t_n) \geq f'(n) \geq f'(x)$ for $x \in [1, n]$, and if the improper integral is convergent then

$$\sum_{n=2}^{\infty}(f^{-1}(f(n)+\epsilon)-n) = \epsilon\sum_{n=2}^{\infty}\frac{1}{f'(t_n)} \leq \epsilon\sum_{n=2}^{\infty}\int_{n-1}^{n}\frac{dx}{f'(x)} = \epsilon\int_1^\infty \frac{dx}{f'(x)},$$

and we may conclude that also the series is convergent.

On the other hand, if the series is convergent, the term $f^{-1}(f(n)+\epsilon) - n = x_n - n \to 0$ and therefore $x_{n+1} - x_n = (x_{n+1} - (n+1)) - (x_n - n) + 1 \to 1$. It follows that for any positive integer n, $0 < x_{n+1} - x_n \leq M$ for some constant $M \geq 1$.
Moreover $f'(t_n) \leq f'(x_n) \leq f'(x)$ for $x \in [x_n, +\infty)$ implies that

$$\frac{M}{\epsilon}\sum_{n=1}^{\infty}(f^{-1}(f(n)+\epsilon)-n) = M\sum_{n=1}^{\infty}\frac{1}{f'(t_n)} \geq \sum_{n=1}^{\infty}\int_{x_n}^{x_{n+1}}\frac{dx}{f'(x)} = \int_{x_1}^{\infty}\frac{dx}{f'(x)},$$

and we have that the improper integral is convergent. □

Remark. If the series

$$\sum_{n=1}^{\infty}(f^{-1}(f(n)+\epsilon)-n)$$

is convergent for one $\epsilon > 0$ then it converges for all $\epsilon > 0$.