Problem 12063
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Proposed by H. Ohtsuka (Japan).

Let p and q be real numbers with $p > 0$ and $q > -p^2/4$. Let $U_0 = 0, U_1 = 1$, and $U_{n+2} = pU_{n+1}+qU_n$ for $n \geq 0$. Calculate

$$\lim_{n\to\infty} \sqrt{U_1^2 + \sqrt{U_2^2 + \sqrt{U_4^2 + \sqrt{\cdots + \sqrt{U_{2^{n-1}}^2}}}}}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Given two real numbers p and q, then U_n is the Lucas sequence of the first kind,

$$U_0 = 0,\ U_1 = 1, \text{ and } U_{n+2} = pU_{n+1} + qU_n \text{ for } n \geq 0,$$

whereas the Lucas sequence of the second kind is defined as

$$V_0 = 2,\ V_1 = p, \text{ and } V_{n+2} = pV_{n+1} + qV_n \text{ for } n \geq 0.$$

Such sequences satisfy the following relations that are generalizations of those between Fibonacci numbers F_n and Lucas numbers L_n (which can be obtained by letting $p = q = 1$):

$$U_{2n} = U_nV_n, \quad 2V_{2n} = V_n^2 + DU_n^2$$

where $D = p^2 + 4q$. Here p and D are positive, and, by induction, U_{2^n}, V_{2^n} are positive too for all $n \geq 0$. Now, for $n \geq 0$, let

$$R_n = \frac{V_{2^n} + \sqrt{D+4}\,U_{2^n}}{2}.$$

Then, $R_n > 0$ and, by the above relations, we have that,

$$\begin{aligned}4R_n^2 &= V_{2^n}^2 + (D+4)U_{2^n}^2 + 2\sqrt{D+4}\,U_{2^n}V_{2^n}\\ &= 2V_{2^{n+1}} + 4U_{2^n}^2 + 2\sqrt{D+4}\,U_{2^{n+1}}\\ &= 4U_{2^n}^2 + V_{2^{n+1}} + 2\sqrt{D+4}\,U_{2^{n+1}} = 4(U_{2^n}^2 + R_{n+1})\end{aligned}$$

that is $R_n = \sqrt{U_{2^n}^2 + R_{n+1}}$. Hence

$$R_0 = \sqrt{U_1^2 + \sqrt{U_2^2 + \sqrt{U_4^2 + \sqrt{\cdots + \sqrt{U_{2^{n-1}}^2 + R_n}}}}} > \sqrt{U_1^2 + \sqrt{U_2^2 + \sqrt{U_4^2 + \sqrt{\cdots + \sqrt{U_{2^{n-1}}^2}}}}}.$$

On the other hand, for $0 < t < 1$,

$$\begin{aligned}tR_0 &= \sqrt{t^2U_1^2 + \sqrt{t^4U_2^2 + \sqrt{t^8U_4^2 + \sqrt{\cdots + \sqrt{t^{2^n}(U_{2^{n-1}}^2 + R_n)}}}}}\\ &< \sqrt{U_1^2 + \sqrt{U_2^2 + \sqrt{U_4^2 + \sqrt{\cdots + \sqrt{t^{2^n}(U_{2^{n-1}}^2 + R_n)}}}}}\\ &< \sqrt{U_1^2 + \sqrt{U_2^2 + \sqrt{U_4^2 + \sqrt{\cdots + \sqrt{U_{2^{n-1}}^2}}}}}\end{aligned}$$

where the last inequality holds eventually because there is some $N(t)$ such that, for all $n \geq N(t)$,

$$t^{2^n}(U_{2^{n-1}}^2 + R_n) < U_{2^{n-1}}^2.$$

This is due to the fact that $\lim_{n\to\infty} V_n/U_n = \sqrt{D}$ and therefore

$$\lim_{n\to\infty} \frac{U_{2^{n-1}}^2}{U_{2^{n-1}}^2 + R_n} = \frac{4}{(\sqrt{D}+\sqrt{D+4})^2} > 0 = \lim_{n\to\infty} t^{2^n}.$$

Hence, for all $t \in (0,1)$, and for all $n \geq N(t)$,

$$tR_0 < \sqrt{U_1^2 + \sqrt{U_2^2 + \sqrt{U_4^2 + \sqrt{\cdots + \sqrt{U_{2^{n-1}}^2}}}}} < R_0$$

which imply that

$$\lim_{n\to\infty} \sqrt{U_1^2 + \sqrt{U_2^2 + \sqrt{U_4^2 + \sqrt{\cdots + \sqrt{U_{2^{n-1}}^2}}}}} = R_0 = \frac{p + \sqrt{p^2+4q+4}}{2}.$$

□

Remark. By letting $p = q = 1$, we have that $U_n = F_n$, the n-th Fibonacci number, and

$$\lim_{n\to\infty} \sqrt{F_1^2 + \sqrt{F_2^2 + \sqrt{F_4^2 + \sqrt{\cdots + \sqrt{F_{2^{n-1}}^2}}}}} = 2.$$