Problem 12060
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Prove

$$\sum_{n=2}^{\infty}\frac{H_nH_{n+1}}{n^3-n}=\frac{5}{2}-\frac{\pi^2}{24}-\zeta(3)$$

where $H_n=\sum_{j=1}^{n}\frac{1}{j}$ *is the nth harmonic number.*

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. It is straightforward to verify that for $n\geq 2$,

$$\begin{aligned}\frac{H_nH_{n+1}}{n^3-n}=&\frac{1}{2}\left(\frac{H_{n+1}^2}{n+1}-\frac{H_n^2}{n}\right)-\frac{1}{2}\left(\frac{H_n^2}{n}-\frac{H_{n-1}^2}{n-1}\right)+\frac{3}{4}\left(\frac{H_{n+1}}{n+1}-\frac{H_n}{n}\right)-\frac{5}{4}\left(\frac{H_n}{n}-\frac{H_{n-1}}{n-1}\right)\\&-\frac{3}{4}\left(\frac{1}{n}-\frac{1}{n-1}\right)-\frac{1}{2}\left(\frac{1}{(n+1)^2}-\frac{1}{n^2}\right)-\frac{1}{4(n+1)^2}-\frac{H_{n+1}}{2(n+1)^2}\end{aligned}$$

(just set $H_{n+1}=H_n+\frac{1}{n+1}$ and $H_{n-1}=H_n-\frac{1}{n}$, expand and simplify). Hence, as $N\to\infty$,

$$\begin{aligned}\sum_{n=2}^{N}\frac{H_nH_{n+1}}{n^3-n}&=-\frac{H_2^2}{4}+\frac{H_1^2}{2}-\frac{3H_2}{8}+\frac{5H_1}{4}+\frac{3}{4}+\frac{1}{8}-\frac{1}{4}\sum_{n=2}^{N}\frac{1}{(n+1)^2}-\frac{1}{2}\sum_{n=2}^{N}\frac{H_{n+1}}{(n+1)^2}+o(1)\\&=\frac{3}{2}-\frac{1}{4}\left(\sum_{n=1}^{N+1}\frac{1}{n^2}-1-\frac{1}{4}\right)-\frac{1}{2}\left(\sum_{n=1}^{N+1}\frac{H_n}{n^2}-H_1-\frac{H_2}{4}\right)+o(1)\to\frac{5}{2}-\frac{\pi^2}{24}-\zeta(3)\end{aligned}$$

where at the last step we used the known results

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\zeta(2)=\frac{\pi^2}{6}\quad\text{and}\quad\sum_{n=1}^{\infty}\frac{H_n}{n^2}=2\zeta(3).$$

□

Appendix. Since $D(\mathrm{Li}_r(z))=\frac{1}{z}\mathrm{Li}_{r-1}(z)$ where $\mathrm{Li}_r(z)=\sum_{k=1}^{\infty}\frac{z^k}{k^r}$, it follows that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{H_n}{n^2}&=\sum_{n=1}^{\infty}\frac{\sum_{j=0}^{n-1}\int_0^1 t^j\,dt}{n^2}=\sum_{n=1}^{\infty}\frac{1}{n^2}\cdot\int_0^1\frac{1-t^n}{1-t}\,dt=\int_0^1\frac{\zeta(2)-\mathrm{Li}_2(t)}{1-t}dt\\&=\left[-\zeta(2)\ln(1-t)-2\mathrm{Li}_3(1-t)+2\mathrm{Li}_2(1-t)\ln(1-t)+\mathrm{Li}_2(t)\ln(1-t)+\ln(t)\ln^2(1-t)\right]_{0^+}^{1^-}\\&=2\mathrm{Li}_3(1)=2\zeta(3)\end{aligned}$$