Problem 12058
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Let b be an integer greater than 1. For a positive integer n, let $u_b(n)$ be the number of nonzero digits in the base b representation of n. Prove that for any positive integers n and k, there exists a positive integer m such that $u_b(mn) = u_b(n) + k$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Given the base b, for $k \geq 1$, let $m_k : \mathbb{N}^+ \to \mathbb{N}^+$ be a function such that

$$u_b(m_k(n)n) = u_b(n) + k.$$

Since for $k \geq 2$,

$$u_b(m_1(m_{k-1}(n)n)m_{k-1}(n)n) = u_b(m_{k-1}(n)n) + 1 = u_b(n) + k - 1 + 1 = u_b(n) + k,$$

we may inductively define $m_k(n) := m_1(m_{k-1}(n)n)m_{k-1}(n)$. Therefore it suffices to show that the function m_1 exists.

We distinguish two cases: $b = 2$ and $b > 2$. Note that for any positive integer n, its base b representation is given by $\sum_{i=0}^{d-1} c_i \cdot b^i$ with $c_i \in \{0, 1, \ldots, b-1\}$ and $d = \lfloor \log_b(n) \rfloor + 1$.

• Case $b = 2$.

Let $0 \leq j < d$ be the smallest index such that $c_j = 1$, then $n = 2^j a$ where a is an odd positive integer. Hence $2^{\varphi(a)} \equiv 1 \pmod{a}$, and

$$\begin{aligned} M &:= 2^{(d+2)\varphi(a)+j-1} + 2^{(d+1)\varphi(a)+j-1} + n - 2^j \equiv 2^j(2^{\varphi(a)-1} + 2^{\varphi(a)-1} - 1) + n \\ &= 2^j(2^{\varphi(a)} - 1) + n \equiv 0 \pmod{n}. \end{aligned}$$

It follows that M is a multiple of n where the last 1-digit of n is removed and two new 1-digits appear (note that $(d+1)\varphi(a) + j - 1 \geq d$. Hence we define $m_1(n) := M/n$.

• Case $b > 2$.

i) Assume that there is $0 \leq j < d$ such that $c_j \geq 2$. Let r, s be two integers such that $r > s + d > 2d$ and $b^r \equiv b^s \pmod{n}$. Then

$$M := b^r + b^{s-j}n - b^s \equiv 0 \pmod{a}$$

that is M is a multiple of n where the digit c_j shifted to the left by $s - j$ units and it is decreased by one (but it still non-zero) and a new digit 1 appears in the leading position. Let $m_1(n) := M/n$.

ii) Assume that there is no $0 \leq j < d$ such that $c_j \geq 2$, that is all the digits of n are 0 or 1. Then $2n$ satisfies i) and it has the same number of nonzero digits of n. Therefore by letting $m_1(n) := 2m_1(2n)$ we get

$$u_b(m_1(n)n) = u_b(m_1(2n)2n) = u_b(2n) + 1 = u_b(n) + 1.$$

□