Problem 12057
(American Mathematical Monthly, Vol.125, August-September 2018)

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(a) Calculate the limit of the sequence defined by $a_1 = 1$, $a_2 = 2$, *and*

$$a_{2k+1} = \frac{a_{2k-1} + a_{2k}}{2} \quad \text{and} \quad a_{2k+2} = \sqrt{a_{2k}a_{2k+1}}$$

for any positive integer k.
(b) Calculate the limit of the sequence defined by $b_1 = 1$, $b_2 = 2$, *and*

$$b_{2k+1} = \frac{b_{2k-1} + b_{2k}}{2} \quad \text{and} \quad b_{2k+2} = \frac{2b_{2k}b_{2k+1}}{b_{2k} + b_{2k+1}}$$

for any positive integer k.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. (a) By using the trigonometric identities

$$\frac{1}{\tan(x)} = \frac{1}{\tan(2x)} + \frac{1}{\sin(2x)} \quad \text{and} \quad 2\sin^2(x) = \sin(2x)\tan(x),$$

it is easy to verify that

$$a_{2k+1} = \frac{\sqrt{3}}{2^k \tan\left(\frac{\pi}{3\cdot 2^k}\right)} \quad \text{and} \quad a_{2k+2} = \frac{\sqrt{3}}{2^k \sin\left(\frac{\pi}{3\cdot 2^k}\right)}$$

satisfy the given recurrence. Hence

$$\lim_{k\to\infty} a_{2k+1} = \lim_{k\to\infty} a_{2k+2} = \lim_{k\to\infty} \frac{\sqrt{3}}{2^k \cdot \frac{\pi}{3\cdot 2^k}} = \frac{3\sqrt{3}}{\pi}$$

and we may conclude that

$$\lim_{k\to\infty} a_k = \frac{3\sqrt{3}}{\pi}.$$

(b) We have that

$$b_n = \prod_{k=1}^{n-1}\left(1 - \frac{1}{2^k}\right)^{(-1)^k}$$

satisfies the given recurrence:

$$\frac{b_{2k-1} + b_{2k}}{2} = \frac{1}{2}\left(\left(1 - \frac{1}{2^{2k}}\right) + 1\right) b_{2k} = \left(1 - \frac{1}{2^{2k+1}}\right) b_{2k} = b_{2k+1}$$

and

$$\frac{b_{2k}^{-1} + b_{2k+1}^{-1}}{2} = \frac{1}{2}\left(\left(1 - \frac{1}{2^{2k+1}}\right) + 1\right) b_{2k+1}^{-1} = \left(1 - \frac{1}{2^{2k+2}}\right) b_{2k+1}^{-1} = b_{2k+2}^{-1}.$$

Therefore

$$\lim_{k\to\infty} b_k = \prod_{k=1}^{\infty}\left(1 - \frac{1}{2^k}\right)^{(-1)^k} = \frac{\prod_{k=0}^{\infty}\left(1 - \frac{1/4}{4^k}\right)}{\prod_{k=0}^{\infty}\left(1 - \frac{1/2}{4^k}\right)} = \frac{(1/4;1/4)_\infty}{(1/2;1/4)_\infty} \approx 1.641632560655$$

where $(a;q)_\infty = \prod_{k=0}^{\infty}(1 - aq^k)$ is the q-Pochhammer symbol. $\square$