Problem 12056
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Let $ABCD$ be a rectangle inscribed in a circle S of radius R, and let P be a point inside S. The lines AP, BP, CP, and DP intersect S a second time at K, L, M, and N, respectively. Prove

$$|AK|^2 + |BL|^2 + |CM|^2 + |DN|^2 \geq \frac{16R^4}{R^2 + |OP|^2}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. It suffices to show that

$$\frac{|AK|^2}{(2R)^2} + \frac{|CM|^2}{(2R)^2} \geq \frac{2R^2}{R^2 + |OP|^2} \quad \text{and} \quad \frac{|BL|^2}{(2R)^2} + \frac{|DN|^2}{(2R)^2} \geq \frac{2R^2}{R^2 + |OP|^2}.$$

We prove only the first inequality since the second one follows by analogous arguments.
Let Q be the orthogonal projection of P onto the diameter AC then the right triangles $\triangle AKC$ and $\triangle AQP$ are similar and therefore

$$\frac{|AK|}{2R} = \frac{|AQ|}{|AP|}.$$

In a similar way,

$$\frac{|CM|}{2R} = \frac{|CQ|}{|CP|}.$$

Moreover, by the Apollonius's theorem applied to the triangle $\triangle APC$ with respect to the median OP:

$$|AP|^2 + |CP|^2 = 2(R^2 + |OP|^2).$$

Thus the desired inequality becomes

$$\frac{|AQ|^2}{|AP|^2} + \frac{|CQ|^2}{|CP|^2} \geq \frac{(2R)^2}{|AP|^2 + |CP|^2},$$

which holds because by Cauchy-Schwarz inequality

$$\left(|AP|^2 + |CP|^2\right) \cdot \left(\frac{|AQ|^2}{|AP|^2} + \frac{|CQ|^2}{|CP|^2}\right) \geq (|AQ| + |CQ|)^2 = (2R)^2.$$

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