Problem 12055
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Proposed by D. E. Knuth (USA).

Let $(a_i)_{i\geq 1}$ be a sequence of nonnegative integers with $a_1 \geq a_2 \geq \dots$ and with finite sum. For a positive integer j, let b_j be the number of indices i such that $a_i \geq j$. Prove that the multisets $\{a_1+1, a_2+2, \dots\}$ and $\{b_1+1, b_2+2, \dots\}$ are equal.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. We show that the property holds by induction with respect to $n \geq 0$, the number of non-zero terms of the sequence $(a_i)_{i\geq 1}$.

i) Base step. If $n = 0$ then

$$(a_i)_{i\geq 1} = (0,0,0,0,\dots) \to \{a_i + i\}_{i\geq 1} = \{1,2,3,4,\dots\}$$
$$(b_j)_{j\geq 1} = (0,0,0,0,\dots) \to \{b_j + j\}_{j\geq 1} = \{1,2,3,4,\dots\}$$

and the multisets are equal.

ii) Inductive step. If $n \geq 1$ then

$$(a_i)_{i\geq 1} = (\underbrace{m, a_2, \dots, a_n}_{n}, 0, \dots) \to \{a_i + i\}_{i\geq 1} = \{\underbrace{\mathbf{m+1}, a_2+2, \dots, a_n+n}_{n}, n+1, n+2, \dots\}$$
$$(b_j)_{j\geq 1} = (\underbrace{n, b_2, \dots, b_m}_{m}, 0, \dots) \to \{b_j + j\}_{j\geq 1} = \{\underbrace{n+1, b_2+2, \dots, b_m+m}_{m}, \mathbf{m+1}, m+2, \dots\}$$

where $a_1 = m \geq 1$.
After removing $\mathbf{m+1}$ and subtracting one to each term, the multisets are equal if and only if

$$\{\underbrace{a_2+1, \dots, a_n+n-1}_{n-1}, n, n+1, \dots\} = \{\underbrace{n, b_2+1, \dots, b_m+m-1}_{m}, m+1, \dots\}.$$

Now the above equality holds by the inductive hypothesis applied to $(a_{i+1})_{i\geq 1} = (\underbrace{a_2, \dots, a_n}_{n-1}, 0, \dots)$: we have that $(b_j)_{j\geq 1} = (\underbrace{n-1, b_2-1, \dots, b_m-1}_{m}, 0, \dots)$ and

$$\{\underbrace{a_2+1, \dots, a_n+n-1}_{n-1}, n, n+1, \dots\} = \{a_{i+1}+i\}_{i\geq 1} = \{b_j+j\}_{j\geq 1} = \{\underbrace{n, b_2+1, \dots, b_m+m-1}_{m}, m+1, \dots\}.$$

$\square$