Problem 12054
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Prove

$$\int_0^1 \frac{\arctan(x)}{x} \ln\left(\frac{1+x^2}{(1-x)^2}\right) dx = \frac{\pi^3}{16}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Let $H_k = \sum_{j=1}^k \frac{1}{j}$ for $k \geq 1$. We have that for $x \in (0,1)$,

$$\begin{aligned}\arctan(x)\ln(1+x^2) &= \frac{i}{2}(\ln(1-ix) - \ln(1+ix))(\ln(1-ix) + \ln(1+ix)) \\ &= \frac{i}{2}(\ln^2(1-ix) - \ln^2(1+ix)) = -\operatorname{Im}\left(\ln^2(1-ix)\right) = -2\operatorname{Im}\left(\sum_{k=1}^\infty \frac{H_k (ix)^{k+1}}{k+1}\right)\end{aligned}$$

where we used the fact that

$$-\ln(1-t) = \sum_{k=1}^\infty \frac{t^k}{k} \implies -\frac{\ln(1-t)}{1-t} = \sum_{k=1}^\infty H_k t^k \implies \ln^2(1-t) = 2\sum_{k=1}^\infty \frac{H_k t^{k+1}}{k+1}.$$

Hence

$$\begin{aligned}\int_0^1 \frac{\arctan(x)\ln(1+x^2)}{x}\, dx &= -2\operatorname{Im}\left(\int_0^1 \sum_{k=1}^\infty \frac{H_k i^{k+1} x^k}{k+1}\, dx\right) \\ &= -2\operatorname{Im}\left(\sum_{k=1}^\infty \frac{H_k i^{k+1}}{k+1}\int_0^1 x^k\, dx\right) = -2\operatorname{Re}\left(\sum_{k=1}^\infty \frac{H_k i^k}{(k+1)^2}\right).\end{aligned}$$

On the other hand,

$$\begin{aligned}\int_0^1 \frac{\arctan(x)\ln(1-x)}{x}\, dx &= \int_0^1 \sum_{k=1}^\infty \frac{(-1)^k x^{2k}\ln(1-x)}{2k+1}\, dx \\ &= \sum_{k=0}^\infty \frac{(-1)^k}{2k+1}\int_0^1 x^{2k}\ln(1-x)\, dx \\ &= \sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)^2}\int_0^1 \ln(1-x)\, d(x^{2k+1}-1) \\ &= -\sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)^2}\int_0^1 \frac{x^{2k+1}-1}{x-1}\, dx \\ &= -\sum_{k=1}^\infty \frac{(-1)^k H_{2k+1}}{(2k+1)^2} = -\operatorname{Re}\left(\sum_{k=0}^\infty \frac{H_{k+1} i^k}{(k+1)^2}\right).\end{aligned}$$

Therefore

$$\begin{aligned}\int_0^1 \frac{\arctan(x)}{x}\ln\left(\frac{1+x^2}{(1-x)^2}\right) dx &= -2\operatorname{Re}\left(\sum_{k=1}^\infty \frac{H_k i^k}{(k+1)^2}\right) + 2\operatorname{Re}\left(\sum_{k=0}^\infty \frac{H_{k+1} i^k}{(k+1)^2}\right) \\ &= 2\operatorname{Re}\left(\sum_{k=0}^\infty \frac{i^k}{(k+1)^3}\right) = 2\operatorname{Im}\left(\sum_{k=1}^\infty \frac{i^k}{k^3}\right) = 2\operatorname{Im}(\operatorname{Li}_3(i)) = \frac{\pi^3}{16}\end{aligned}$$

because, by a known identity for the trilogarithm Li_3,

$$2\operatorname{Im}(\operatorname{Li}_3(i)) = \operatorname{Im}\left(\operatorname{Li}_3(z) - \operatorname{Li}_3(1/z)\right)_{z=i} = -\frac{1}{6}\operatorname{Im}\left(\ln^3(-z) + \pi^2\ln(-z)\right)_{z=i} = \frac{\pi^3}{16}.$$

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