Problem 12051
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Prove

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{1}{4^n(2n+1)^3}=\frac{\pi^3}{48}+\frac{\pi\ln^2(2)}{4}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. We have that

$$\int_0^1 x^{2n}\ln(x)\,dx=\left[x^{2n+1}\left(\frac{\ln(x)}{(2n+1)}-\frac{1}{(2n+1)^2}\right)\right]_0^1=-\frac{1}{(2n+1)^2}$$

and, for $x\in[0,1]$,

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{x^{2n}}{4^n(2n+1)}=\frac{\arcsin(x)}{x}.$$

Hence

$$\begin{aligned}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{1}{4^n(2n+1)^3}&=-\int_0^1\frac{\arcsin(x)\ln(x)}{x}\,dx\\&=-\left[\arcsin(x)\,\frac{\ln^2(x)}{2}\right]_0^1+\frac{1}{2}\int_0^1\frac{\ln^2(x)}{\sqrt{1-x^2}}\,dx\\&=\frac{1}{2}\int_0^1\frac{\ln^2(x)}{\sqrt{1-x^2}}\,dx=\frac{1}{2}\int_0^{\pi/2}\ln^2(\sin(t))\,dt=\frac{\pi^3}{48}+\frac{\pi\ln^2(2)}{4}\end{aligned}$$

where the last integral is known.
For the sake of completeness, we give here a short proof. Since $\ln^n(1+z)$ is holomorphic in the unit disc and it is zero for $z=0$, it follows that

$$0=\int_{-\pi}^{\pi}\ln^n(1+e^{i\theta})\,d\theta=\int_{-\pi}^{\pi}(\ln(2\cos(\theta/2)+i\theta/2)^n\,d\theta=2\int_{-\pi/2}^{\pi/2}(\ln(2\cos(t))+it)^n\,dt.$$

By taking the real part we get

$$\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\binom{n}{2k}\int_0^{\pi/2}t^{2k}\ln^{n-2k}(2\cos t)\,dt=0.$$

For $n=1$ and for $n=2$, we get

$$\int_0^{\pi/2}\ln(\cos t)\,dt=-\frac{\pi}{2}\ln(2)\quad,\quad\int_0^{\pi/2}\ln^2(2\cos t)\,dt=\int_0^{\pi/2}t^2\,dt=\frac{\pi^3}{24}.$$

Hence

$$\begin{aligned}\int_0^{\pi/2}\ln^2(\sin(t))\,dt&=\int_0^{\pi/2}\ln^2(\sin(\pi/2-t))\,dt=\int_0^{\pi/2}\ln^2(\cos(t))\,dt\\&=\int_0^{\pi/2}\ln^2(2\cos(t))\,dt-2\ln(2)\int_0^{\pi/2}\ln(\cos(t))\,dt-\frac{\pi\ln^2(2)}{2}\\&=\frac{\pi^3}{24}+\pi\ln^2(2)-\frac{\pi\ln^2(2)}{2}=\frac{\pi^3}{24}+\frac{\pi\ln^2(2)}{2}.\end{aligned}$$

□