Problem 12049
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Proposed by Z. K. Silagadze (Russia).

For all nonnegative integers m and n with $m \leq n$, prove

$$\sum_{k=m}^{n} \frac{(-1)^{k+m}}{2k+1} \binom{n+k}{n-k} \binom{2k}{k-m} = \frac{1}{2n+1}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. We have to show that

$$S_n = \sum_{k=m}^{n} F(n,k) = 1$$

where

$$F(n,k) = \frac{(-1)^{k+m}(2n+1)}{2k+1} \binom{n+k}{n-k} \binom{2k}{k-m}.$$

Let

$$G(n,k) = \frac{2(-1)^{k+m}(m^2-k^2)(n+1)}{((n+1)^2-m^2)(n+1+k)} \binom{n+1+k}{n+1-k} \binom{2k}{k-m}$$

then (F,G) is Wilf-Zeilberger pair and the following identity holds

$$F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k).$$

Hence

$$\begin{aligned} S_{n+1} - S_n &= \sum_{k=m}^{n+1} (F(n+1,k) - F(n,k)) + F(n,n+1) \\ &= \sum_{k=m}^{n+1} (G(n,k+1) - G(n,k)) + F(n,n+1) \\ &= G(n,n+2) - G(n,m) + F(n,n+1) = 0 - 0 + 0 = 0 \end{aligned}$$

which implies inductively that for all $n \geq m$, $S_n = S_m = F(m,m) = 1$. $\square$