Problem 12048
(American Mathematical Monthly, Vol.125, June-July 2018)

Proposed by J. C. Lagarias (USA).

Call an integer a special Carmichael number if it can be written as, $(6k+1)(12k+1)(18k+1)$ *for some integer* k*, with each of* $6k+1$, $12k+1$*, and* $18k+1$ *being prime. Call an integer a taxicab number if it can be written as the sum of two positive integer cubes in two different ways. Show that 1729 is the only positive integer that is both a special Carmichael number and a taxicab number.*

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Let n be a positive integer which can be written as the sum of the cubes of two positive integers x and y then there exists a positive divisor d of n such that $d = x + y$ and

$$n = x^3 + y^3 = x^3 + (d-x)^3 = d(3x^2 - 3dx + d^2) \implies x^2 - dx + \frac{d^3 - n}{3d} = 0.$$

Since this quadratic equation has two positive integer solutions x and y then $d^3 - n > 0$ and $3d\Delta = 3d^3 - 4(d^3 - n) = 4n - d^3 \geq 0$, that is

$$n < d^3 \leq 4n. \tag{$\star$}$$

Here n is the product of three primes $p < q < r$ with $p = 6k + 1 \geq 7$, $q = 2p - 1$ and $r = 3p - 2$. Now, for each divisor $d \in \{1, p, q, r, pq, pr, qr, pqr\}$, we check whether the necessary condition $(\star)$ is satisfied.

1) If $d \in \{1, p\}$ then $n = pqr > p^3 \geq d^3$ and $(\star)$ does not hold.

2) If $d = q$ then $(\star)$ is satisfied if and only if $pr < q^2 \leq 4pr$ that is

$$p(3p-2) < (2p-1)^2 \leq 4p(3p-2)$$

which holds for $p \geq 2$.

3) If $d = r$ then $(\star)$ is satisfied if and only if $pq < r^2 \leq 4pq$ that is

$$4pq - r^2 = 4p(2p-1) - (3p-2)^2 = -p^2 + 8p - 4 = 12 - (p-4)^2 \geq 0$$

which holds only for $2 \leq p \leq 7$.

4) If $d \in \{pq, pr, qr, pqr\}$ then $d^3 \geq p^3q^3 > 4pqr = 4n$ and $(\star)$ does not hold because for $p \geq 2$,

$$p^2q^2 - 4r = p^2(2p-1)^2 - 4(3p-2) = 4p^4 - 4p^3 + p^2 - 12p + 8 = 4p^3(p-1) + (p-6)^2 - 28 \geq 32 - 28 > 0.$$

Hence, in order to have at least two different ways to write n as the sum of two positive integer cubes, it is necessary that both 2) and 3) hold, which happens just for $p = 7$. Moreover, in such case,

$$n = 7 \cdot 13 \cdot 19 = 1729 = 1^3 + 12^3 = 9^3 + 10^3.$$

where $1^3 + 12^3$ and $9^3 + 10^3$ are obtained respectively for $d = q = 13$ and for $d = r = 19$. $\square$