Problem 12046
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Proposed by M. Omarjee (France).

Suppose that $f : [0,1] \to \mathbb{R}$ has a continuous and nonnegative third derivative, and suppose $\int_0^1 f(x)dx = 0$. Prove

$$10\int_0^1 x^3 f(x)\,dx + 6\int_0^1 x f(x)\,dx \geq 15\int_0^1 x^2 f(x)\,dx.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Let $P(x) = (x(1-x))^3$, then $P^{(k)}(0) = P^{(k)}(1) = 0$ for $k = 0, 1, 2$. Hence, by integration by parts,

$$\int_0^1 P^{(3)}(x)f(x)\,dx = -\int_0^1 P^{(2)}(x)f'(x)\,dx = \int_0^1 P^{(1)}(x)f''(x)\,dx = -\int_0^1 P(x)f'''(x)\,dx \leq 0$$

where, at the last step, we used the fact that P and f''' are nonnegative in $[0,1]$. It is easy to verify that $P^{(3)}(x) = -12(10x^3 - 15x^2 + 6x - 1/2)$ and therefore

$$\int_0^1 (10x^3 - 15x^2 + 6x - 1/2)f(x)\,dx \geq 0$$

and, recalling that $\int_0^1 f(x)dx = 0$, we get

$$10\int_0^1 x^3 f(x)\,dx + 6\int_0^1 x f(x)\,dx \geq 15\int_0^1 x^2 f(x)\,dx.$$

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