Problem 12045
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Proposed by O. Furdui and A. Sintamarian (Romania).

Prove that

$$\sum_{n=1}^{\infty}(-1)^{n-1}\left(n\sum_{k=n+1}^{\infty}\frac{1}{k^2}-1\right)=\frac{\pi^2}{16}-\frac{\ln(2)}{2}-\frac{1}{2}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. By Stolz-Cesaro theorem,

$$\lim_{N\to\infty}\frac{\zeta(2)-\sum_{k=1}^{N}1/k^2}{1/N}=\lim_{N\to\infty}\frac{1/(N+1)^2}{1/N-1/(N+1)}=1 \implies \zeta(2)-\sum_{k=1}^{N}\frac{1}{k^2}=\frac{1}{N}+o(1/N).$$

For $N>k\geq 1$, then

$$\sum_{n=k+1}^{N}(-1)^n=\frac{(-1)^N-(-1)^k}{2},\qquad \sum_{n=k+1}^{N}(-1)^n n=\frac{(-1)^N N-(-1)^k k}{2}+\frac{(-1)^N-(-1)^k}{4}.$$

Hence, as $N\to\infty$,

$$\begin{aligned}
\sum_{n=1}^{N}(-1)^{n-1}\left(n\sum_{k=n+1}^{\infty}\frac{1}{k^2}-1\right)&=\sum_{n=1}^{N}(-1)^{n-1}\left(n\zeta(2)-n\sum_{k=1}^{n}\frac{1}{k^2}-1\right)\\
&=-\zeta(2)\sum_{n=1}^{N}(-1)^n n+\sum_{n=1}^{N}(-1)^n n\sum_{k=1}^{n}\frac{1}{k^2}+\sum_{n=1}^{N}(-1)^n\\
&=-\zeta(2)\sum_{n=1}^{N}(-1)^n n+\sum_{n=1}^{N}(-1)^n+\sum_{k=1}^{N}\frac{(-1)^k}{k}+\sum_{k=1}^{N}\frac{1}{k^2}\sum_{n=k+1}^{N}(-1)^n n\\
&=-\zeta(2)\left(\frac{(-1)^N N}{2}+\frac{(-1)^N-1}{4}\right)+\frac{(-1)^N-1}{2}+\sum_{k=1}^{N}\frac{(-1)^k}{k}\\
&\quad+\sum_{k=1}^{N}\frac{1}{k^2}\left(\frac{(-1)^N N-(-1)^k k}{2}+\frac{(-1)^N-(-1)^k}{4}\right)\\
&=-(-1)^N\left(\frac{N}{2}+\frac{1}{4}\right)\left(\zeta(2)-\sum_{k=1}^{N}\frac{1}{k^2}\right)+\frac{\zeta(2)}{4}+\frac{(-1)^N-1}{2}\\
&\quad+\frac{1}{2}\sum_{k=1}^{N}\frac{(-1)^k}{k}-\frac{1}{4}\sum_{k=1}^{N}\frac{(-1)^k}{k^2}\\
&=-(-1)^N\left(\frac{N}{2}+\frac{1}{4}\right)\left(\frac{1}{N}+o(1/N)\right)+\frac{\zeta(2)}{4}+\frac{(-1)^N-1}{2}\\
&\quad-\frac{\ln(2)}{2}+\frac{\zeta(2)}{8}+o(1)\\
&=\frac{3\zeta(2)}{8}-\frac{\ln(2)}{2}-\frac{1}{2}+o(1)\to\frac{\pi^2}{16}-\frac{\ln(2)}{2}-\frac{1}{2}.
\end{aligned}$$

□