Problem 12041
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Proposed by R. Stanley (USA).

For a positive integer c, let $\nu_p(c)$ denote the largest integer d such that p^d divides c. Let

$$H_m = \prod_{i=0}^{m}\prod_{j=0}^{m}\binom{i+j}{i}.$$

For $n \geq 1$, prove

$$\nu_p(H_{p^n-1}) = \frac{1}{2}\left(\left(n - \frac{1}{p-1}\right)p^{2n} + \frac{p^n}{p-1}\right).$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. We have that

$$\begin{aligned}
\nu_p(H_m) &= \sum_{i=0}^{m}\sum_{j=0}^{m}(\nu_p((i+j)!) - \nu_p(i!) - \nu_p(j!)) \\
&= \sum_{i=0}^{m}(i+1)\nu_p(i!) + \sum_{i=0}^{m-1}(m-i)\nu_p((i+m+1)!) - 2(m+1)\sum_{i=0}^{m}\nu_p(i!) \\
&= \sum_{i=0}^{m}(m-i)\left(\nu_p((i+m+1)!) - \nu_p(i!)\right) - (m+1)\sum_{i=0}^{m}\nu_p(i!).
\end{aligned}$$

Let p be a prime. Then by Legendre Theorem,

$$\nu_p(N!) = \sum_{k\geq 1}\left\lfloor \frac{N}{p^k}\right\rfloor = \frac{N - s_p(N)}{p-1}$$

where $s_p(N)$ is the sum of all the digits in the expansion of N in base p.
For $m = p^n - 1$,

$$\begin{aligned}
\sum_{i=0}^{p^n-1}(p^n-1-i)\left(\nu_p((i+p^n)!) - \nu_p(i!)\right) &= \sum_{k=1}^{n}\sum_{i=0}^{p^n-1}(p^n-1-i)\left(\left\lfloor\frac{i+p^n}{p^k}\right\rfloor - \left\lfloor\frac{i}{p^k}\right\rfloor\right) \\
&= \sum_{k=1}^{n}\sum_{i=0}^{p^n-1}(p^n-1-i)p^{n-k} = \frac{p^n(p^n-1)^2}{2(p-1)}.
\end{aligned}$$

Moreover, since $\sum_{i=0}^{p^n-1} s_p(i) = \frac{1}{2}np^n(p-1)$, it follows that

$$\sum_{i=0}^{p^n-1}\nu_p(i!) = \sum_{i=0}^{p^n-1}\frac{i - s_p(i)}{p-1} = \frac{p^n(p^n-1)}{2(p-1)} - \frac{np^n}{2}.$$

Hence

$$\begin{aligned}
\nu_p(H_{p^n-1}) &= \frac{p^n(p^n-1)^2}{2(p-1)} - p^n\left(\frac{p^n(p^n-1)}{2(p-1)} - \frac{np^n}{2}\right) \\
&= \frac{1}{2}\left(\left(n - \frac{1}{p-1}\right)p^{2n} + \frac{p^n}{p-1}\right).
\end{aligned}$$

□