Problem 12039
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Proposed by S. Silwal (USA).

Let G be a graph with an even number of vertices. Show that there are two vertices in G with an even number of common neighbors.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Suppose to the contrary that every two vertices have an odd number of common neighbors. Let $v \in V(G)$, let $N(v)$ be set of neighbors of v and let $U(v) = V(G) - N(v) - \{v\}$ be the set of non-neighbors of v. If $x \in N(v)$ then $xy \in E(G)$ with $y \in N(v)$ if and only if $y \in N(v) \cap N(x)$. Hence the degree of x inside the graph induced by $N(v)$ is $|N(v) \cap N(x)|$ which is odd by the hypothesis. Since the sum of the degrees of all vertices of a graph is even, it follows that $|N(v)|$ is even and therefore the degree of v in G is even (actually we showed that any vertex of G has even degree).

Let $y \in U(v)$ then $x \in N(v)$ and $xy \in E(G)$ if and only if $x \in N(v) \cap N(y)$ which implies, by the hypothesis, that the number of edges between $N(v)$ and y is odd. Since the degree of y in G is even, it follows that the degree of y in the graph induced by $U(v)$ is odd. Again, since the sum of the degrees of all vertices of a graph is even, we may conclude that $|U(v)|$ is even.

Finally we have that $V(G) = 1 + |N(v)| + |U(v)|$ is odd contradicting the fact that the total number of vertices of G is even. $\square$