Problem 12034
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Let N be any natural number that is not a multiple of 10. *Prove that there is a multiple of N each of whose digits in base* 10 *is* 1, 2, 3, 4, *or* 5.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. We first show three claims.

i) 2^d has a multiple $m_2(d)$ of d digits containing only digits 1 or 2.

Let $m_2(1) = 2$ and

$$m_2(d+1) = \begin{cases} 2 \cdot 10^d + m_2(d) & \text{if } m_2(d)/2^d \equiv 0 \pmod 2, \\ 1 \cdot 10^d + m_2(d) & \text{if } m_2(d)/2^d \equiv 1 \pmod 2. \end{cases}$$

Hence if $m_2(d)/2^d \equiv r \pmod 2$ then

$$m_2(d+1) = (2-r)10^d + m_2(d) = (2-r)10^d + (2k+r)2^d = 2^{d+1}\left((5^d + k) - \frac{(5^d-1)}{2} r\right)$$

and, since 2 divides $5^d - 1$, it follows that 2^{d+1} divides $m_2(d+1)$.

ii) 5^d has a multiple $m_5(d)$ of d digits containing only digits 1, 2, 3, 4, or 5.

Let $m_5(1) = 5$ and

$$m_5(d+1) = \begin{cases} 5 \cdot 10^d + m_5(d) & \text{if } 3^d\, m_5(d)/5^d \equiv 0 \pmod 5, \\ 4 \cdot 10^d + m_5(d) & \text{if } 3^d\, m_5(d)/5^d \equiv 1 \pmod 5, \\ 3 \cdot 10^d + m_5(d) & \text{if } 3^d\, m_5(d)/5^d \equiv 2 \pmod 5, \\ 2 \cdot 10^d + m_5(d) & \text{if } 3^d\, m_5(d)/5^d \equiv 3 \pmod 5, \\ 1 \cdot 10^d + m_5(d) & \text{if } 3^d\, m_5(d)/5^d \equiv 4 \pmod 5. \end{cases}$$

Hence if $3^d\, m_5(d)/5^d \equiv r \pmod 5$ then

$$3^d m_5(d+1) = 3^d(5-r)10^d + 3^d m_5(d) = (5-r)30^d + (5k+r)5^d = 5^{d+1}\left((6^d + k) - \frac{(6^d-1)}{5} r\right)$$

and, since 5 divides $6^d - 1$, it follows that 5^{d+1} divides $m_5(d+1)$.

iii) Let a be a positive integer such that $\gcd(a, 10) = 1$. Then for any $d \geq 1$ there is a positive integer n such that

$$R_n(d) := \sum_{k=0}^{n-1} 10^{kd} = \overbrace{1\underbrace{00\ldots01}_{d}\ldots\underbrace{00\ldots01}_{d}\underbrace{00\ldots01}_{d}}^{n \text{ digits } 1}$$

is a multiple a.

Since the sequence $\{R_n(d)\}_n$ is infinite, there are two distinct positive integers $n_1 < n_2$ such that

$$R_{n_2}(d) \equiv R_{n_1}(d) \pmod a$$

which implies that a divides

$$R_{n_2}(d) - R_{n_1}(d) = 10^{n_1 d} R_{n_2 - n_1}(d).$$

Since $\gcd(a, 10) = 1$ it follows that a divides $R_{n_2-n_1}(d)$.

Finally, let N be a positive integer which is not a multiple of 10. Then N can be written as $2^d \cdot a$ or $5^d \cdot a$ with $d \geq 0$ and $\gcd(a, 10) = 1$. If $d = 0$ then, by iii), $R_n(1)$ is a multiple of N which contains only the digit 1. If $d \geq 1$ and $N = 2^d \cdot a$ then, by i) and ii), $R_n(d)m_2(d)$ is a multiple of N which contains only the digits 1 or 2. If $d \geq 1$ and $N = 5^d \cdot a$ then, by i) and iii), $R_n(d)m_5(d)$ is a multiple of N which contains only the digits 1, 2, 3, 4, or 5. $\square$