Problem 12032
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Proposed by D. Galante and A. Plaza (Spain).

For a positive integer n, compute

$$\sum_{j=0}^{n}\sum_{k=j}^{n}(-1)^{k-j}\binom{k}{2j}\binom{n}{k}2^{n-k}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. We have that

$$\begin{aligned}\sum_{j=0}^{n}\sum_{k=j}^{n}(-1)^{k-j}\binom{k}{2j}\binom{n}{k}2^{n-k} &= \sum_{k=0}^{n}(-1)^{k}\binom{n}{k}2^{n-k}\sum_{j=0}^{\lfloor k/2\rfloor}(-1)^{j}\binom{k}{2j}\\ &= \sum_{k=0}^{n}(-1)^{k}\binom{n}{k}2^{n-k}\mathrm{Re}\left((1+i)^{k}\right)\\ &= \mathrm{Re}\left(\sum_{k=0}^{n}\binom{n}{k}2^{n-k}(-1-i)^{k}\right)\\ &= \mathrm{Re}\left((2-1-i)^{n}\right)=\mathrm{Re}\left((1-i)^{n}\right)\\ &= 2^{n/2}\mathrm{Re}\left(e^{-\frac{in\pi}{4}}\right)=2^{n/2}\cos\left(\frac{n\pi}{4}\right)\end{aligned}$$

which is always an integer number (see sequence A146559 in OEIS). □