Problem 12031
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Proposed by O. Furdui (Romania).

(a) Prove

$$\int_0^1 \int_0^1 \left\{ \frac{x}{1-xy} \right\} dx\, dy = 1 - \gamma,$$

where $\{a\}$ denotes the fractional part of a, and γ is Euler's constant.

(b) Let k be a nonnegative integer. Prove

$$\int_0^1 \int_0^1 \left\{ \frac{x}{1-xy} \right\}^k dx\, dy = \int_0^1 \left\{ \frac{1}{x} \right\}^k dx.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. (a) follows from (b). For $k = 1$, by letting $t = 1/x$ we have,

$$\begin{aligned}\int_0^1 \int_0^1 \left\{ \frac{x}{1-xy} \right\} dx\, dy &= \int_0^1 \left\{ \frac{1}{x} \right\} dx = \int_1^\infty \frac{\{t\}}{t^2} dt = \sum_{n=1}^\infty \int_n^{n+1} \frac{t-n}{t^2} dt \\ &= \lim_{N\to\infty} \sum_{n=1}^{N-1} \left(\ln(n+1) - \ln(n) - \frac{1}{n+1} \right) = \lim_{N\to\infty} (\ln(N) - H_N + 1) = 1 - \gamma\end{aligned}$$

where $H_N := \sum_{n=1}^N 1/n$ is the N-th harmonic number.

As regards (b), let $t = \frac{1}{x} - y$ then $x = \frac{1}{t+y}$, $\left| \frac{\partial(x,y)}{\partial(t,y)} \right| = \frac{1}{(t+y)^2}$,

$$\begin{aligned}\int_0^1 \int_0^1 \left\{ \frac{x}{1-xy} \right\}^k dx\, dy &= \int_{t=0}^1 \int_{y=1-t}^1 \left\{ \frac{1}{t} \right\}^k \frac{dt\, dy}{(y+t)^2} + \int_{t=1}^\infty \int_{y=0}^1 \left\{ \frac{1}{t} \right\}^k \frac{dt\, dy}{(y+t)^2} \\ &= \int_0^1 \left\{ \frac{1}{t} \right\}^k \left(1 - \frac{1}{t+1} \right) dt + \int_1^\infty \frac{1}{t^k} \left(\frac{1}{t} - \frac{1}{t+1} \right) dt \\ &= \int_0^1 \left\{ \frac{1}{t} \right\}^k dt - \int_0^1 \left\{ \frac{1}{t} \right\}^k \frac{dt}{t+1} + \int_1^\infty \frac{dt}{t^{k+1}(t+1)} \\ &= \int_0^1 \left\{ \frac{1}{t} \right\}^k dt\end{aligned}$$

where the last equality holds because by letting $s = 1/t$,

$$\begin{aligned}\int_0^1 \left\{ \frac{1}{t} \right\}^k \frac{dt}{t+1} &= \int_1^\infty \frac{\{s\}^k}{(1/s+1)s^2} ds = \sum_{n=1}^\infty \int_n^{n+1} \frac{(s-n)^k}{s(s+1)} ds \\ &= \lim_{N\to\infty} \sum_{n=1}^{N-1} \int_0^1 s^k \left(\frac{1}{s+n} - \frac{1}{s+1+n} \right) ds \\ &= \lim_{N\to\infty} \int_0^1 s^k \sum_{n=1}^{N-1} \left(\frac{1}{s+n} - \frac{1}{s+1+n} \right) ds \\ &= \lim_{N\to\infty} \int_0^1 s^k \left(\frac{1}{s+1} - \frac{1}{s+N} \right) = \int_0^1 \frac{s^k}{s+1} ds \\ &= \int_1^\infty \frac{(1/t)^k}{(1/t+1)t^2} dt = \int_1^\infty \frac{dt}{t^{k+1}(t+1)}.\end{aligned}$$

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