Problem 12029
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For $a>0$, evaluate

$$\lim_{n\to\infty}\prod_{k=1}^{n}\left(a+\frac{k}{n}\right).$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Let $f_n(x)=\ln\left(a+\frac{x}{n}\right)$ then

$$\sum_{k=1}^{n} f_n(k)=\int_0^n f_n(x)dx+\frac{f_n(n)-f_n(0)}{2}+\frac{1}{2}\int_0^n f_n''(x)p(\{x\})dx, \tag{1}$$

where $p(t)=t-t^2$. Therefore

$$\begin{aligned}\sum_{k=1}^{n}\ln\left(a+\frac{k}{n}\right)&=n\ln\left(\frac{(a+1)^{a+1}}{a^a e}\right)+\ln\left(\sqrt{\frac{a+1}{a}}\right)-\frac{1}{2}\int_0^n\frac{p(\{x\})}{(na+x)^2}dx\\&=n\ln\left(\frac{(a+1)^{a+1}}{a^a e}\right)+\ln\left(\sqrt{\frac{a+1}{a}}\right)+O(1/n)\end{aligned}$$

because $0\le p(\{x\})\le 1/4$, and

$$0\le\int_0^n\frac{p(\{x\})}{(na+x)^2}dx\le\frac{1}{4}\int_0^n\frac{dx}{(na+x)^2}=\frac{1}{4na(a+1)}.$$

Finally, as $n\to+\infty$,

$$\prod_{k=1}^{n}\left(a+\frac{k}{n}\right)=\left(\frac{(a+1)^{a+1}}{a^a e}\right)^n\cdot\sqrt{\frac{a+1}{a}}\cdot\exp\left(O(1/n)\right)\to\begin{cases}+\infty & \text{if } a>c,\\ \sqrt{\frac{c+1}{c}} & \text{if } a=c,\\ 0 & \text{if } 0<a<c,\end{cases}$$

where $c\approx 0.542211420$ is the unique positive real number such that $(c+1)^{c+1}=c^c e$. □

Proof of (1). If $f\in C^1[0,n]$ then for $k=0,\dots n-1$, by integrating by parts twice,

$$\begin{aligned}\int_k^{k+1} f(x)dx&=\int_0^1 f(x+k)d(x-1/2)=\left[f(x+k)\left(x-\frac{1}{2}\right)\right]_0^1+\frac{1}{2}\int_0^1 f'(x+k)d(x-x^2)\\&=\frac{f(k+1)+f(k)}{2}+\frac{1}{2}\left[f'(x+k)\left(x-x^2\right)\right]_0^1-\frac{1}{2}\int_0^1 f''(x+k)(x-x^2)dx\\&=\frac{f(k+1)+f(k)}{2}-\frac{1}{2}\int_k^{k+1} f''(x)p(\{x\})dx.\end{aligned}$$

Then, after summing, we obtain

$$\begin{aligned}\int_0^n f(x)dx&=\frac{1}{2}\sum_{k=0}^{n-1} f(k+1)+\frac{1}{2}\sum_{k=0}^{n-1} f(k)-\frac{1}{2}\int_0^n f''(x)p(\{x\})dx\\&=\sum_{k=1}^{n} f(k)-\frac{f(n)-f(0)}{2}-\frac{1}{2}\int_0^n f''(x)p(\{x\})dx.\end{aligned}$$