Problem 12027
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Proposed by A. Hannan (India).

Let ABC be a triangle with circumradius R and inradius r. Let D, E, and F be the points where the incircle of ABC touches BC, CA, and AB, respectively, and let X, Y, and Z be the second points of intersection between the incircle of ABC and AD, BE, and CF, respectively. Prove

$$\frac{|AX|}{|XD|} + \frac{|BY|}{|YE|} + \frac{|CZ|}{|ZF|} = \frac{R}{r} - \frac{1}{2}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Let $a = |BC|$, $b = |AC|$, $c = |AB|$. Let p be the semiperimeter and let S be the area of the triangle ABC. We have that $|AF| = p - a$ and, by the law of cosines,

$$|AD|^2 = |AC|^2 + |CD|^2 - 2|AC||CD|\cos(C) = b^2 + (p-c)^2 - 2b(p-c) \cdot \frac{a^2 + b^2 - c^2}{2ab}.$$

Hence, by the power of a point theorem, $|AX| \cdot |AD| = |AF|^2$, and

$$\frac{|AX|}{|XD|} = \frac{|AX|}{|AD| - |AX|} = \left(\frac{|AD|}{|AX|} - 1\right)^{-1} = \left(\frac{|AD|^2}{|AF|^2} - 1\right)^{-1} = \frac{a(p-a)}{4(p-b)(p-c)}$$

In a similar way, we find $\frac{|BY|}{|YE|}$, $\frac{|CZ|}{|ZF|}$, and

$$\begin{aligned}\frac{|AX|}{|XD|} + \frac{|BY|}{|YE|} + \frac{|CZ|}{|ZF|} + \frac{1}{2} &= \frac{a(p-a)}{4(p-b)(p-c)} + \frac{b(p-b)}{4(p-a)(p-c)} + \frac{c(p-c)}{4(p-c)(p-a)} + \frac{1}{2} \\ &= \frac{ap(p-a)^2 + bp(p-b)^2 + cp(p-c)^2 + 2p(p-a)(p-b)(p-c)}{4S^2} \\ &= \frac{abcp}{4S^2} = \frac{R}{r}.\end{aligned}$$