Problem 12026
(American Mathematical Monthly, Vol.125, February 2018)

Proposed by M. Bataille (France).

For $n \in \mathbb{N}$, *let* $H_n = \sum_{k=1}^n 1/k$ *and*

$$S_n = \sum_{k=1}^{n} \frac{(-1)^{n-k}}{k} \sum_{j=1}^{k} H_j$$

Find $\lim_{n\to\infty} S_n/\ln(n)$ *and* $\lim_{n\to\infty}(S_{2n} - S_{2n-1})$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. It easy to verify by induction that $\sum_{j=1}^k H_j = k(H_k - 1) + H_k$, hence

$$S_{2n} = \sum_{k=1}^{2n}(-1)^k H_k - \sum_{k=1}^{2n}(-1)^k + \sum_{k=1}^{2n}(-1)^k \frac{H_k}{k} = \frac{H_n}{2} + \sum_{k=1}^{2n}(-1)^k \frac{H_k}{k}$$

and

$$S_{2n-1} = -\sum_{k=1}^{2n-1}(-1)^k H_k - \sum_{k=1}^{2n-1}(-1)^k - \sum_{k=1}^{2n-1}(-1)^k \frac{H_k}{k} = H_{2n} - \frac{H_n}{2} - 1 - \sum_{k=1}^{2n-1}(-1)^k \frac{H_k}{k}.$$

Therefore

$$S_{2n} - S_{2n-1} = H_n + 1 - H_{2n} - \frac{H_{2n}}{2n} + 2\sum_{k=1}^{2n}(-1)^k \frac{H_k}{k}.$$

Now, as N goes to infinity,

$$\begin{aligned}
\sum_{k=1}^{N}(-1)^k \frac{H_k}{k} &= \sum_{k=1}^{N} \frac{(-1)^k}{k^2} + \sum_{k=2}^{N}(-1)^k \frac{H_{k-1}}{k} \\
&= \sum_{k=1}^{N} \frac{(-1)^k}{k^2} + \frac{1}{2}\sum_{k=2}^{N}\sum_{j=1}^{k-1} \frac{(-1)^k}{k}\left(\frac{1}{j} + \frac{1}{k-j}\right) \\
&= \sum_{k=1}^{N} \frac{(-1)^k}{k^2} + \frac{1}{2}\sum_{j=1}^{N-1}\sum_{k=j+1}^{N} \frac{(-1)^k}{j(k-j)} \\
&= \sum_{k=1}^{N} \frac{(-1)^k}{k^2} + \frac{1}{2}\sum_{j=1}^{N-1} \frac{(-1)^j}{j}\sum_{k'=1}^{N-j} \frac{(-1)^{k'}}{k'} \\
&= \sum_{k=1}^{N} \frac{(-1)^k}{k^2} + \frac{1}{2}\left(\sum_{j=1}^{N-1} \frac{(-1)^j}{j}\right)^2 - \frac{1}{2}\sum_{j=1}^{N-1} \frac{(-1)^j}{j}\sum_{k'=N-j+1}^{N-1} \frac{(-1)^{k'}}{k'} \to -\frac{\pi^2}{12} + \frac{\ln^2(2)}{2}
\end{aligned}$$

where

$$\left|\sum_{j=1}^{N-1} \frac{(-1)^j}{j}\sum_{k'=N-j+1}^{N-1} \frac{(-1)^{k'}}{k'}\right| \leq \frac{1}{\sqrt{N}} \to 0.$$

Hence, since $H_n = \ln(n) + \gamma + o(1)$,

$$\lim_{n\to\infty}(S_{2n} - S_{2n-1}) = \ln(n) + 1 - \ln(2n) - \frac{\pi^2}{6} + \ln^2(2) + o(1) \to \ln^2(2) - \ln(2) + 1 - \frac{\pi^2}{6} \approx -0.85760324596$$

and

$$\lim_{n\to\infty} \frac{S_n}{\ln(n)} = \lim_{n\to\infty} \frac{H_n}{2\ln(n)} = \frac{1}{2}.$$

□