Problem 12024
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Let x, y, and z be positive real numbers satisfying $xyz = 1$. Prove

$$(x^{10} + y^{10} + z^{10})^2 \geq 3(x^{13} + y^{13} + z^{13}).$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. We will show the following stronger inequality: for any real numbers a, b, and c,

$$(a^5 + b^5 + c^5)^2 \geq 3abc(a^7 + b^7 + c^7). \tag{1}$$

Then, the given inequality follows by letting $a = x^2$, $b = y^2$ and $c = z^2$,

$$\begin{aligned}(x^{10} + y^{10} + z^{10})^2 &\geq 3(xyz)^2(x^{14} + y^{14} + z^{14}) = \frac{3}{2}\sum_{\text{sym}} x^{16}y^2z^2 \\ &\geq \frac{3}{2}\sum_{\text{sym}} x^{46/3}y^{7/3}z^{7/3} = 3(xyz)^{7/3}(x^{13} + y^{13} + z^{13}) = 3(x^{13} + y^{13} + z^{13})\end{aligned}$$

where in the second line we applied the Muirhead's inequality.

Proof of the inequality (1) (note that it has appeared as Monthly problem 11543 in 2010).
Without loss of generality we may assume that $a \geq b \geq c$. Let

$$\begin{aligned}T_a &:= \frac{1}{2}(a^7 + b^7 + c^7)(a + b + c) - b^3c^3(b + c)^2, \\ T_b &:= \frac{1}{2}(a^7 + b^7 + c^7)(a + b + c) - c^3a^3(c + a)^2, \\ T_c &:= \frac{1}{2}(a^7 + b^7 + c^7)(a + b + c) - a^3b^3(a + b)^2.\end{aligned}$$

Then $T_a \geq T_b \geq T_c$ and

$$\begin{aligned}T_b + T_c &= (a^7 + b^7 + c^7)(a + b + c) - c^3a^3(c + a)^2 - a^3b^3(a + b)^2 \\ &= a^8 + b^8 + c^8 + \sum_{\text{sym}} a^7b - c^3a^3(c + a)^2 - a^3b^3(a + b)^2 \\ &\geq a^8 + b^8 + c^8 + \sum_{\text{sym}} a^5b^3 - c^3a^3(c + a)^2 - a^3b^3(a + b)^2 \\ &= (a^5 + b^5 + c^5)(a^3 + b^3 + c^3) - c^3a^3(c + a)^2 - a^3b^3(a + b)^2 \\ &= (a^4 - (b^4 + c^4))^2 + b^3c^3(b - c)^2 \geq 0\end{aligned}$$

where in the third line we applied again the Muirhead's inequality.
Hence $T_a \geq T_b \geq 0$, $(a - c)^2 \geq (a - b)^2$, and finally it is straightforward to verify that

$$\begin{aligned}(a^5 + b^5 + c^5)^2 - 3abc(a^7 + b^7 + c^7) &= (b - c)^2T_a + (a - c)^2T_b + (a - b)^2T_c \\ &\geq (b - c)^2T_a + (a - b)^2(T_b + T_c) \geq 0.\end{aligned}$$

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