Problem 12023
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Let α be a positive real number. Prove

$$\int_0^{\pi} x^{\alpha-2}\sin(x)\,dx \geq \frac{\pi^{\alpha}(\alpha+6)}{\alpha(\alpha+2)(\alpha+3)}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Since the function $f(x) := (1-\cos(x))/x^2$ is concave in $(0,\pi/2]$ (see below), it follows that the tangent line at $\pi/2$ stays above the graph of f: for $x \in (0,\pi/2]$,

$$\frac{1-\cos(x)}{x^2} = f(x) \leq f'(\pi/2)(x-\pi/2) + f(\pi/2) = \frac{2(6-\pi)}{\pi^2} - \frac{4(4-\pi)x}{\pi^3}.$$

Hence, for $s \in [0,1/2]$,

$$\cos(\pi s) \geq 1 - 2(6-\pi)s^2 + 4(4-\pi)s^3 \geq 1 - 6s^2 + 4s^3,$$

and

$$\sin(\pi t) = \pi\int_0^t \cos(\pi s)ds \geq \pi\int_0^t (1-6s^2+4s^3)ds = \pi(t-2t^3+t^4) := p(t).$$

Moreover, the symmetries $\sin(\pi t) = \sin(\pi(1-t))$ and $p(t) = p(1-t)$ imply that the above inequality holds in $[0,1]$. Then

$$\begin{aligned}\int_0^{\pi} x^{\alpha-2}\sin(x)\,dx &= \pi^{\alpha-1}\int_0^1 t^{\alpha-2}\sin(\pi t)\,dt\\ &\geq \pi^{\alpha-1}\int_0^1 t^{\alpha-2}p(t)\,dt\\ &= \pi^{\alpha}\int_0^1 (t^{\alpha-1}-2t^{\alpha+1}+t^{\alpha+2})\,dt\\ &= \pi^{\alpha}\left(\frac{1}{\alpha}-\frac{2}{\alpha+2}+\frac{1}{\alpha+3}\right) = \frac{\pi^{\alpha}(\alpha+6)}{\alpha(\alpha+2)(\alpha+3)}.\end{aligned}$$

□

Proof of the concavity of $f(x) = (1-\cos(x))/x^2$ in $(0,\pi/2]$. Let $g(x) := x^4 f''(x)$ then

$$\begin{aligned}g(x) &= (x^2-6)\cos(x) - 4x\sin(x) + 6,\\ g'(x) &= -(x^2-2)\sin(x) - 2x\cos(x),\\ g''(x) &= -x^2\cos(x).\end{aligned}$$

Now for $x \in [0,\pi/2]$, $g''(x) \leq 0$ and, together with $g''(0) = 0$, it follows that $g'(x) \leq 0$. Similarly, $g(x) \leq 0$ and we may conclude that f si concave in $(0,\pi/2]$.