Problem 12022
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Proposed by M. Merca (Romania).

Let n be a positive integer, and let x be a real number not equal to -1 or 1. Prove

$$\sum_{k=0}^{n-1} \frac{(1-x^n)(1-x^{n-1})\cdots(1-x^{n-k})}{1-x^{k+1}} = n$$

and

$$\sum_{k=0}^{n-1} (-1)^k \frac{(1-x^n)(1-x^{n-1})\cdots(1-x^{n-k})}{1-x^{k+1}} \cdot x^{\binom{n-1-k}{2}} = n x^{\binom{n}{2}}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. More generally, we show by induction that that for any positive integer n and any complex number z such that $|z| \neq 1$, then

$$F_n(z) = \sum_{k=0}^{n-1} \frac{\prod_{j=n-k}^{n}(1-z^j)}{1-z^{k+1}} = n.$$

We have that $F_1(z) = \frac{1-z}{1-z} = 1$, and by using the identity

$$1 - z^{n+1} = (1-z^{k+1}) - (1-z^{k+1})(1-z^{n-k}) + (1-z^{n-k}),$$

we have that for $n \geq 1$,

$$\begin{aligned} F_{n+1}(z) &= \sum_{k=0}^{n} \frac{(1-z^{n+1})\prod_{j=n+1-k}^{n}(1-z^j)}{1-z^{k+1}} \\ &= \sum_{k=0}^{n} \prod_{j=n+1-k}^{n} (1-z^j) - \sum_{k=0}^{n-1} \prod_{j=n-k}^{n} (1-z^j) + \sum_{k=0}^{n-1} \frac{\prod_{j=n-k}^{n}(1-z^j)}{1-z^{k+1}} \\ &= 1 + \sum_{k'=0}^{n-1} \prod_{j=n-k'}^{n} (1-z^j) - \sum_{k=0}^{n-1} \prod_{j=n-k}^{n} (1-z^j) + n = 1+n, \end{aligned}$$

where an empty product is 1. As regards the second identity, we have that

$$\begin{aligned} nz^{\binom{n}{2}} = z^{\binom{n}{2}} F_n(1/z) &= z^{\binom{n}{2}} \sum_{k=0}^{n-1} \frac{\prod_{j=n-k}^{n}(1-(1/z)^j)}{1-(1/z)^{k+1}} \\ &= z^{\binom{n}{2}} \sum_{k=0}^{n-1} (-1)^k \frac{\prod_{j=n-k}^{n}(1-z^j)}{1-z^{k+1}} \cdot \frac{z^{k+1}}{\prod_{j=n-k}^{n} z^j} \\ &= \sum_{k=0}^{n-1} (-1)^k \frac{\prod_{j=n-k}^{n}(1-z^j)}{1-z^{k+1}} \cdot z^{\binom{n}{2}+k+1-\sum_{j=n-k}^{n} j} \\ &= \sum_{k=0}^{n-1} (-1)^k \frac{\prod_{j=n-k}^{n}(1-z^j)}{1-z^{k+1}} \cdot z^{\binom{n-1-k}{2}} \end{aligned}$$

and we are done. □