Problem 12019
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Find all positive integers n such that $(2^n-1)(5^n-1)$ is a perfect square.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. For $n=1$, $(2^1-1)(5^1-1)=2^2$. For $n\geq 2$, we claim that $(2^n-1)(5^n-1)$ is not a perfect square.

For $n\geq 1$, $(2^n-1)(5^n-1)$ is a perfect square if and only if $2^n-1=du^2$ and $5^n-1=dv^2$ with u, v are positive integers and d is positive square-free integer (an integer which is divisible by no perfect square other than 1). We consider four different cases.

1) If $n=4j+1$ with $j\geq 1$ then

$$(2^n-1)(5^n-1)=(2\cdot 16^j-1)(5\cdot 625^j-1)\equiv(-1)(5-1)\equiv 12 \pmod{16}$$

which is not a quadratic residue modulo 16.

2) If $n=4j+3$ with $j\geq 0$ then

$$(2^n-1)(5^n-1)=(8\cdot 16^j-1)(5^n-1)\equiv(3-1)(-1)\equiv 3 \pmod{5}$$

which is not a quadratic residue modulo 5.

3) If $n=2j$ with $j\geq 1$ and $d=1$ then $2^n-1=u^2$ and $5^n-1=v^2$ imply

$$1=2^n-u^2=(2^j-u)(2^j+u)>2 \quad\text{and}\quad 1=5^n-v^2=(5^j-v)(5^j+v)>5$$

which do not hold.

4) If $n=2j$ with $j\geq 1$ and $d>1$ then $2^n-1=du^2$ and $5^n-1=dv^2$ the it follows that d, u are odd, and v is even. Note that $(x,y)=(2^j,u)$ and $(x,y)=(5^j,v)$ are solutions of the Pell's equation (d is not a perfect square),

$$x^2-dy^2=1.$$

Let (x_1,y_1) be the fundamental solution, then any other solution (x_k,y_k) is such that

$$x_k+y_k\sqrt{d}=(x_1+y_1\sqrt{d})^k$$

for some positive integer k. We have that

$$x_{2k}+y_{2k}\sqrt{d}=(x_k+y_k\sqrt{d})^2=x_k^2+dy_k^2+2x_ky_k\sqrt{d}=(2x_k^2-1)+2x_ky_k\sqrt{d}$$

which implies that x_{2k} is odd and y_{2k} is even. Moreover

$$\begin{aligned}x_{2k+1}+y_{2k+1}\sqrt{d}&=(x_{2k}+y_{2k}\sqrt{d})(x_1+y_1\sqrt{d})\\&=(x_{2k}x_1+dy_{2k}y_1)+(x_{2k}y_1+y_{2k}x_1)\sqrt{d}\end{aligned}$$

which implies that x_{2k+1} has the same parity of x_1, and y_{2k+1} has the same parity of y_1. Hence $(2^j,u)=(x_i,y_i)$ for some odd integer i and $(5^j,v)=(x_l,y_l)$ for some even integer $l=2k$. Finally

$$5^j=x_{2k}=2x_k^2-1 \implies x_k^2\equiv 3 \pmod{5}$$

which does not hold because 3 is not a quadratic residue modulo 5.

□