Problem 12016
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Proposed by H. Ohtsuka (Japan), and R. Tauraso (Italy).

For non-negative integers m, n, r, and s, prove that

$$\sum_{k=0}^{s}\binom{m+r}{n-k}\binom{r+k}{k}\binom{s}{k}=\sum_{k=0}^{r}\binom{m+s}{n-k}\binom{s+k}{k}\binom{r}{k}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. We have to show that $A(m,n,r,s)=A(m,n,s,r)$ where

$$A(m,n,r,s):=\sum_{k=0}^{s}\binom{m+r}{n-k}\binom{r+k}{k}\binom{s}{k}.$$

Since

$$\binom{m+1+r}{n-k}=\binom{m+r}{n-k}+\binom{m+r}{n-1-k},$$

we have that the following recurrence holds

$$A(m+1,n,r,s)=A(m,n,r,s)+A(m,n-1,r,s).$$

Therefore, by using induction with respect to m, it suffices to prove the base case for $m=0$: $A(0,n,r,s)=A(0,n,s,r)$. Consider the generating function in three variables

$$F(x,u,v):=\sum_{n,s,r\geq 0}A(0,n,r,s)x^n u^s v^r,$$

then

$$\begin{aligned}
F(x,u,v)&=\sum_{r,s\geq 0}\left(\sum_{k\geq 0}\binom{r+k}{k}\binom{s}{k}x^k\right)\cdot\left(\sum_{k\geq 0}\binom{r}{k}x^k\right)\cdot u^s v^r\\
&=\sum_{r\geq 0}\left(\sum_{k\geq 0}\binom{r+k}{k}x^k\cdot\sum_{s\geq 0}\binom{s}{k}u^s\right)\cdot(1+x)^r v^r\\
&=\sum_{r\geq 0}\left(\sum_{k\geq 0}\binom{r+k}{k}x^k\cdot\frac{u^k}{(1-u)^{k+1}}\right)\cdot((1+x)v)^r\\
&=\frac{1}{1-u}\sum_{r\geq 0}\left(\sum_{k\geq 0}\binom{r+k}{k}\left(\frac{xu}{1-u}\right)^k\right)\cdot((1+x)v)^r\\
&=\frac{1}{1-u}\sum_{r\geq 0}\frac{((1+x)v)^r}{\left(1-\frac{xu}{1-u}\right)^{r+1}}\\
&=\frac{1}{1-(1+x)u}\sum_{r\geq 0}\left(\frac{(1-u)(1+x)v}{1-(1+x)u}\right)^r\\
&=\frac{1}{1-(1+x)u}\cdot\frac{1}{1-\frac{(1-u)(1+x)v}{1-(1+x)u}}\\
&=\frac{1}{(1+x)(vu-(v+u))+1},
\end{aligned}$$

which is symmetric with respect to interchange of u and v. Hence

$$A(0,n,r,s)=[x^n u^s v^r]F(x,u,v)=[x^n u^s v^r]F(x,v,u)=A(0,n,s,r)$$

and we are done. □