Problem 12015
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Proposed by Dao Thanh Oai (Vietnam).

Let ABC be a triangle, let G be its centroid, and let D, E, and F be the midpoints of BC, CA, and AB, respectively. For any point P in the plane of ABC, prove

$$|PA| + |PB| + |PC| \leq 2(|PD| + |PE| + |PF|) + 3|PG|,$$

and determine when equality holds.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Let P, A, B, and C be points in the complex plane, then $2D = B + C$, $2E = C + A$, $2F = A + B$, and $3G = A + B + C$. Let $P - A = x$, $P - B = y$, $P - C = z$ then

$$2(P - D) = y + z, \quad 2(P - E) = z + x, \quad 2(P - F) = x + y, \quad 3(P - G) = x + y + z.$$

Hence the given inequality is equivalent to

$$|x| + |y| + |z| \leq |x + y| + |y + z| + |z + x| + |x + y + z|. \tag{1}$$

In order to show that

$$|x + y + z| + |x + y| + |y + z| + |z + x| - |x| - |y| - |z| \geq 0,$$

we multiply both sides by $|x| + |y| + |z| - |x + y + z| > 0$ (we assume that the triangle is not degenerate), and, by noting that

$$|x + y + z|^2 + |x|^2 + |y|^2 + |z|^2 = |x + y|^2 + |y + z|^2 + |z + x|^2,$$

after some algebraic manipulation the product reduces to

$$\begin{aligned}&(|x| + |y| - |x + y|)(|x + y + z| + |x + y| - |z|)\\ &+ (|y| + |z| - |y + z|)(|x + y + z| + |y + z| - |x|)\\ &+ (|z| + |x| - |z + x|)(|x + y + z| + |z + x| - |y|) \geq 0\end{aligned}$$

which holds because by the triangle inequality in each of the three terms, both factors are not negative.

The equality is satisfied only when each of the three terms vanishes. Since $|z| + |w| = |z + w|$ iff z, w are on the same ray from the origin, it follows that the equality holds iff $x + y + z = 0$, i.e. $P = G$ (again we are assuming that the triangle is not degenerate). $\square$

Remark. Inequality (1) can be obtained from Hlawka's inequality:

$$|u + v| + |v + w| + |w + u| \leq |u| + |v| + |w| + |u + v + w|.$$

Let $u = x + y - z$, $v = x + z - y$, $w = y + z - x$, then

$$\begin{aligned}2|x| + 2|y| + 2|z| &= |u + v| + |v + w| + |w + u|\\ &\leq |u| + |v| + |w| + |u + v + w| = |x + y - z| + |x + z - y| + |y + z - x| + |x + y + z|\\ &\leq |x + y| + |z| + |x + z| + |y| + |y + z| + |x| + |x + y + z|\end{aligned}$$

which is equivalent to (1).