Problem 12014
(American Mathematical Monthly, Vol.125, January 2018)

Proposed by O. Furdui (Romania).

Let a, b, c, and d be real numbers with $bc > 0$. Calculate

$$\lim_{n\to\infty} \begin{bmatrix} \cos(a/n) & \sin(b/n) \\ \sin(c/n) & \cos(d/n) \end{bmatrix}^n .$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Let

$$M_n = \begin{bmatrix} \cos(a/n) & \sin(b/n) \\ \sin(c/n) & \cos(d/n) \end{bmatrix}$$

then its characteristic polynomial is $x^2 - \operatorname{tr}(M_n)x + \det(M_n)$ with two distinct eigenvalues (for n sufficiently large)

$$\lambda_n = \frac{1}{2}\left(\operatorname{tr}(M_n) + \sqrt{\Delta_n}\right) \qquad \text{and} \qquad \mu_n = \frac{1}{2}\left(\operatorname{tr}(M_n) - \sqrt{\Delta_n}\right)$$

where $\Delta_n = (\cos(a/n) - \cos(d/n))^2 + 4\sin(b/n)\sin(c/n) > 0$ (note that $bc > 0$) with eigenvectors respectively

$$u_n = \begin{bmatrix} \lambda_n - \cos(d/n) \\ \sin(c/n) \end{bmatrix} \qquad \text{and} \qquad v_n = \begin{bmatrix} \mu_n - \cos(d/n) \\ \sin(c/n) \end{bmatrix}.$$

We have that

$$\operatorname{tr}(M_n) = 2 + O(1/n^2) \qquad \text{and} \qquad \Delta_n = \frac{4bc}{n^2} + O(1/n^4),$$

which imply

$$\lambda_n = 1 + \frac{\sqrt{bc}}{n} + O(1/n^2) \qquad \text{and} \qquad \mu_n = 1 - \frac{\sqrt{bc}}{n} + O(1/n^2).$$

Let $P_n = [u_n|v_n]$ then $P_n^{-1}M_nP_n = \operatorname{diag}(\lambda_n, \mu_n)$ and as n goes to infinity,

$$\begin{aligned} M_n^n &= P_n \operatorname{diag}(\lambda_n^n, \mu_n^n) P_n^{-1} \\ &\sim \frac{1}{n}\begin{bmatrix} \sqrt{bc} & -\sqrt{bc} \\ c & c \end{bmatrix} \cdot \begin{bmatrix} e^{\sqrt{bc}} & 0 \\ 0 & e^{-\sqrt{bc}} \end{bmatrix} \cdot \frac{n}{2}\begin{bmatrix} 1/\sqrt{bc} & 1/c \\ -1/\sqrt{bc} & 1/c \end{bmatrix} \\ &\sim \frac{1}{2}\begin{bmatrix} \sqrt{bc} & -\sqrt{bc} \\ c & c \end{bmatrix} \begin{bmatrix} e^{\sqrt{bc}}/\sqrt{bc} & e^{\sqrt{bc}}/c \\ -e^{-\sqrt{bc}}/\sqrt{bc} & e^{-\sqrt{bc}}/c \end{bmatrix} \\ &\to \begin{bmatrix} \cosh\sqrt{bc} & b \cdot \dfrac{\sinh\sqrt{bc}}{\sqrt{bc}} \\ c \cdot \dfrac{\sinh\sqrt{bc}}{\sqrt{bc}} & \cosh\sqrt{bc} \end{bmatrix}. \end{aligned}$$

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