Problem 12012
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Proposed by O. Furdui and Alina Sintamarian (Romania).

Let k be a nonnegative integer. Find the set of real numbers x for which the power series

$$\sum_{n=k}^{\infty} \binom{n}{k} \left(e - 1 - \frac{1}{1!} - \frac{1}{2!} - \cdots - \frac{1}{n!}\right) x^n$$

converges, and determine the sum.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Let

$$F_k(x) := \sum_{n=k}^{\infty} \binom{n}{k} \left(e - \sum_{j=0}^{n} \frac{1}{j!}\right) x^n.$$

Note that by Taylor's theorem, for $n \geq k$, there is $t \in (0,1)$ such that

$$\left| \binom{n}{k} \left(e - \sum_{j=0}^{n} \frac{1}{j!}\right) x^n \right| \leq \binom{n}{k} \frac{e^t}{(n+1)!} |x|^n \leq \frac{e|x|^k}{k!(n+1)} \cdot \frac{|x|^{n-k}}{(n-k)!}$$

and, by the Weierstrass M-Test, the power series $F_k(x)$ converges for any real number x and it converges uniformly on any bounded subset in $\mathbb{R}$ (so we can interchange summation with differentiation). Now, for $k = 0$,

$$\begin{aligned} F_0(x) &= \sum_{n=0}^{\infty} \left(e - \sum_{j=0}^{n} \frac{1}{j!}\right) x^n = \sum_{n=0}^{\infty} \left(\sum_{j=n+1}^{\infty} \frac{1}{j!}\right) x^n \\ &= \sum_{j=1}^{\infty} \frac{1}{j!} \left(\sum_{n=0}^{j-1} x^n\right) = \sum_{j=1}^{\infty} \frac{1}{j!} \frac{1-x^j}{1-x} = \begin{cases} \dfrac{e - e^x}{1-x} & \text{if } x \neq 1, \\ e & \text{if } x = 1. \end{cases} \end{aligned}$$

By using the general Leibniz rule for the k-th derivative of a product, we get

$$\begin{aligned} D^{(k)} \left(\frac{e - e^x}{1-x}\right) &= e D^{(k)}((1-x)^{-1}) - \sum_{j=0}^{k} \binom{k}{j} D^{(k-j)}(e^x) \cdot D^{(j)}((1-x)^{-1}) \\ &= \frac{k! e}{(1-x)^{k+1}} - e^x \sum_{j=0}^{k} \binom{k}{j} \frac{j!}{(1-x)^{j+1}}. \end{aligned}$$

Hence

$$\begin{aligned} F_k(x) &= \frac{x^k}{k!} \sum_{n=k}^{\infty} \left(e - \sum_{j=0}^{n} \frac{1}{j!}\right) D^{(k)}(x^n) = \frac{x^k}{k!} D^{(k)}(F_0(x)) \\ &= \begin{cases} \dfrac{x^k}{(1-x)^{k+1}} \left(e - e^x \displaystyle\sum_{j=0}^{k} \frac{(1-x)^j}{j!}\right) & \text{if } x \neq 1, \\ \dfrac{e}{(k+1)!} & \text{if } x = 1. \end{cases} \end{aligned}$$

This is a more compact formula

$$F_k(x) = x^k e^x \sum_{j=0}^{\infty} \frac{(1-x)^j}{(j+k+1)!}.$$

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